

# Mediators in Position Auctions

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## ABSTRACT

A mediator is a reliable entity, which can play on behalf of agents in a given game. A mediator however can not enforce the use of its services, and each agent is free to participate in the game directly. In this paper we introduce a study of mediators for games with incomplete information, and apply it to the context of position auctions, a central topic in electronic commerce. VCG position auctions, which are currently not used in practice, possess some nice theoretical properties, such as the optimization of social surplus and having dominant strategies. These properties may not be satisfied by current position auctions and their variants. We therefore concentrate on the search for mediators that will allow to transform current position auctions into VCG position auctions. We require that accepting the mediator services, and reporting honestly to the mediator, will form an ex post equilibrium, which satisfies the following rationality condition: an agent's payoff can not be negative regardless of the actions taken by the agents who did not choose the mediator's services, or by the agents who report false types to the mediator. We prove the existence of such desired mediators for the next-price (Google-like) position auctions, as well as for a richer class of position auctions, including all  $k$ -price position auctions,  $k > 1$ . For  $k=1$ , the self-price position auction, we show that the existence of such mediator depends on the tie breaking rule used in the auction.

## Categories and Subject Descriptors

J.4 [Social and Behavioral Sciences]: [Economics]; I.2 [Artificial Intelligence]: Distributed Artificial Intelligence—*Multiagent systems*

## General Terms

Economics, Theory

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## Keywords

Auction, Mediator, Equilibrium

## 1. INTRODUCTION

Consider an interaction in a multi-agent system, in which every player holds some private information, which is called the player's type. For example, in an auction interaction the type of a player is its valuation, or, in more complex auctions, its valuation function. Every player has a set of actions, and a strategy of a player is a function that maps each of its possible types to an action. This interaction is modeled as a game with incomplete information. This game is called a Bayesian game, when a commonly known probability measure on the profiles of types is added to the system. Otherwise it is called a pre-Bayesian game. In this paper we deal only with pre-Bayesian games. The leading solution concept for pre-Bayesian games is the ex post equilibrium: A profile of strategies, one for each player, such that no player has a profitable deviation independently of the types of the other players. Consider the following simple example of a pre-Bayesian game, which possesses an ex post equilibrium. The game is denoted by  $\mathbb{H}$ .

	$a$	$b$		$a$	$b$	
$a$	5, 2	3, 0		$a$	2, 2	0, 0
$b$	0, 0	4, 2	A	$b$	3, 3	5, 2

At the game  $\mathbb{H}$  there are two players. Both players can choose among two actions:  $a$  and  $b$ . The column player, player 2, has a private type,  $A$  or  $B$  (player 1 has only one possible type). A strategy of player 1 is  $g_1$ , where  $g_1 = a$  or  $g_1 = b$ . A strategy of player 2 is a function  $g_2 : \{A, B\} \rightarrow \{a, b\}$ . That is, player 2 has 4 strategies. In this game the strategy profile  $(g_1, g_2)$  is an ex post equilibrium, where  $g_1 = b$  and  $g_2(A) = b, g_2(B) = a$ .

Unfortunately, pre-Bayesian games do not, in general, possess ex post equilibria, even if we allow mixed strategies. In order to address this problem we suggest in this paper the use of *mediators*. A mediator is a reliable entity that can interact with the players and perform on their behalf actions in a given game. However, a mediator can not enforce behavior. Indeed, an agent is free to participate in the game without the help of the mediator. The mediator's behavior on behalf of the agents that give it the right of play is pre-specified, and is conditioned on information the agents would provide to the mediator. This notion is highly natural; in many systems there is some form of reliable party or

administrator that can be used as a mediator. The simplest form of a mediator discussed in the game theory literature is captured by the notion of correlated equilibrium [1]. This notion was generalized to communication equilibrium by [5, 15]. Another type of mediators is discussed in [13]. However, in all these settings the mediator can not perform actions on behalf of the agents that allow it to do so. Mediators that can obtain the "right of play" but can not enforce the use of their services have been already defined and discussed for games with complete information in [14].<sup>1</sup> The topic of mediators for games with complete information has been further generalized and analyzed in [16]. In this paper we introduce another use of mediators, in establishing behaviors which are stable against unilateral deviations in games with incomplete information. Notice that we assume that the multi-agent interaction (formalized as a game) is given, and all the mediator can do is to perform actions on behalf of the agents that explicitly allow it to do so.<sup>2</sup>

In order to illustrate the power of mediators for games with incomplete information consider the following pre-Bayesian game  $\mathbb{G}$  that does not possess an ex post equilibrium. In  $\mathbb{G}$ , the column player has two possible types:  $A$  and  $B$ .

	$a$	$b$	
$a$	5, 2	3, 0	
$b$	0, 0	2, 2	
	A		

	$a$	$b$	
$a$	2, 2	0, 0	
$b$	3, 0	5, 2	
	B		

A mediator for  $\mathbb{G}$  should specify the actions it will choose on behalf of the players that give it the right to play. If player 2 wants to give the mediator the right to play it should also report a type. Consider the following mediator:

If both players give the mediator the right of play, then the mediator will play on their behalf  $(a, a)$  if player 2 reports  $A$  and  $(b, b)$  if player 2 reports  $B$ . If only player 1 gives the mediator the right of play then the mediator will choose  $a$  on his behalf. If only player 2 gives the mediator the right of play, the mediator will choose action  $a$  (resp.  $b$ ) on its behalf, if  $B$  (resp.  $A$ ) has been reported.

The mediator generates a new pre-Bayesian game, which is called the *mediated game*. In the mediated game player 1 has three actions: Give the mediator the right of play, denoted by  $m$ , or play directly  $a$  or  $b$ . Player 2 has four actions:  $m - A, m - B, a, b$ , where  $m - A$  ( $m - B$ ) means reporting  $A$  ( $B$ ) to the mediator and give it the right of play. The mediated game is described in the following figure:

	$m - A$	$m - B$	$a$	$b$	
$m$	5, 2	2, 2	5, 2	3, 0	
$a$	3, 0	5, 2	5, 2	3, 0	
$b$	2, 2	0, 0	0, 0	2, 2	
	A				

<sup>1</sup>For games with complete information the main interest is in leading agents to behaviors, which are stable against deviations by coalitions. A special case of mediators was already discussed in [8]. In this paper the authors discussed mediators for a two-person game, which is known to the players but not to the mediators, and they looked for Nash equilibrium in the new game generated by the mediator.

<sup>2</sup>This natural setting is different from the one discussed in the classical theories of implementation and mechanism design, where a designer designs a new game from scratch in order to yield some desired behavior.

	$m - A$	$m - B$	$a$	$b$	
$m$	2, 2	5, 2	2, 2	0, 0	
$a$	0, 0	2, 2	2, 2	0, 0	
$b$	5, 2	3, 0	3, 0	5, 2	

B

It is now easy to verify that giving the mediator the right of play, and reporting truthfully, is an ex-post equilibrium at the mediated game. That is,  $(f_1, f_2)$  is an ex post equilibrium, where  $f_1 = m$ , and  $f_2(A) = m - A, f_2(B) = m - B$ .

The aim of this paper is twofold. We introduce mediators for games with incomplete information, and apply them in the context of position auctions. Our choice of position auctions as the domain of application is not an accident; indeed, position auctions have become a central issue in advertisement and the selection of appropriate position auctions for that task is a subject of considerable amount of study [17, 3, 9, 4].<sup>3</sup> Current position auctions however do not possess ex-post equilibrium, i.e. solutions which are stable against unilateral deviations regardless of the agents' private information, nor guarantee optimal social surplus. In contrast, in the VCG position auction, which is currently not used in practice, there is a truth-revealing ex post equilibrium, which yields optimal surplus. We therefore suggest the use of mediators in order to attempt and implement the output of the VCG position auction, by transforming other (and in particular current) position auctions into a VCG position auction.<sup>4</sup> More specifically, the mediated game will have an ex post equilibrium, which generates the outcome of the VCG position auction. One such mediator has already been discussed for other purposes in the literature: An English auction type of algorithm was constructed in [3] that takes as an input the valuations of the players and outputs bids for the next-price position auction. It was proved there that reporting the true type to this algorithm by each player forms an ex post equilibrium, which generates the VCG outcome. In our language this algorithm can be almost considered as a mediator for the next-price position auction that implement the VCG outcome function. What is missing, is a component that punishes players who send their bids directly to the auctioneer, and a proof that using the mediator services and reporting the true type by each player is an ex post equilibrium in the mediated game defined by the algorithm and by the additional component. A mediator may generate a desired outcome by punishing the players who do not use its services with very high bids by the players that use its services. We believe that such mediators are not realistic, and therefore we concentrate on the search for *valid* mediators that generate an ex post equilibrium and satisfy the additional rationality condition: an agent's payoff can not be negative regardless of the actions taken by the agents who did not choose the mediator's services, or agents who report false types to the mediator. We prove the existence of such desired mediators for the next-price (Google-like) position auctions<sup>5</sup>, as well as for a richer class of position auctions, including all  $k$ -price position auctions,  $k > 1$ . For  $k=1$ , the self-price position auction, we show that the ex-

<sup>3</sup>See also [12], where position auctions are titled ad auctions.

<sup>4</sup>In general, except for the VCG position auction we do not expect position auctions to possess an ex post equilibrium (see Footnote 7).

<sup>5</sup>Our proof uses an algorithm, which is different from the algorithm in [3] discussed earlier.

istence of such mediator depends on the tie breaking rule used in the auction.

Mediators in one-item auctions (in particular first price and second price auctions) have been already discussed in [6, 11, 2]; however they all used a Bayesian model. Position auctions are a restricted type of general pre-Bayesian games. In this conference version we make the formal definition of mediators and implementing by mediation for the special case of position auctions, and only in the full version we present the general theory of mediators for pre-Bayesian games. Most of the proofs are omitted from this conference version.

## 2. POSITION AUCTIONS

In a position auction there is a seller who sells a finite number of positions  $j \in K = \{1, \dots, m\}$ . There is a finite number of (potential) bidders  $i \in N = \{1, \dots, n\}$ . We assume that there are more bidders than positions, i.e.  $n > m$ . The positions are sold for a fixed period of time. For each position  $j$  there is a commonly-known number  $\alpha_j > 0$ , which is interpreted as the expected number of visitors at that position.  $\alpha_j$  is called the *click-through rate* of position  $j$ . We assume that  $\alpha_1 > \alpha_2 > \dots > \alpha_m > 0$ . If  $i$  holds a position then every visitor to this position gives  $i$  a revenue of  $v_i > 0$ , where  $v_i$  is called the *valuation* of  $i$ . The set of possible valuations of  $i$  is  $V_i = (0, \infty)$ .

We assume that the players' utility functions are quasi-linear. That is, if player  $i$  is assigned to position  $j$  and pays  $p_i$  per click then his utility is  $\alpha_j(v_i - p_i)$ .

Every player is required to submit a bid,  $b_i \in B_i = [0, \infty)$ . We assume that bidding 0 is a symbol for non-participation. Therefore, a player with a bid 0 is not assigned to any position, and it pays 0.

In all position auctions we consider, the player with the highest positive bid receives the first position, the player with the second highest positive bid receives the second position, and so on. It is useful to define for every position auction two dummy positions  $m+1$  and  $-1$ , which more than one player may be "assigned" to. All players, who participate in the auction but do not get a position in  $K$  are assigned to position  $m+1$  and all players who choose not to participate are assigned to position  $-1$ . We also define  $\alpha_{m+1} = \alpha_{-1} = 0$ .

An assignment of players to positions is called an *allocation*. Hence, an allocation is a vector  $\mathbf{s} = (s_1, s_2, \dots, s_n)$  with  $s_i \in K \cup \{-1, m+1\}$  such that if  $s_i \in K$  then  $s_i \neq s_l$  for every  $l \neq i$ ;  $s_i$  is the position of player  $i$ . Given the above, a position auction is defined by its tie breaking rule, which determines the allocation in case of ties, and by its payment scheme. These are discussed below.

### 2.1 Tie breaking rules

In practice, the most commonly used tie breaking rule is the First-Arrival rule: if a set of players submit the same bid, their priority in receiving the positions is determined by the times their bids were recorded; An earlier bid receives a higher priority. In auction theory this tie breaking rule is typically modelled by assuming that the auctioneer is using a random priority rule. More specifically, let  $\Gamma$  be the set of all permutations,  $\gamma = (\gamma_1, \dots, \gamma_n)$  of  $N$ . Every such  $\gamma$  defines a priority rule as follows:  $i$  has a higher priority than  $k$  if and only if  $\gamma_i < \gamma_k$ . Every vector of bids  $\mathbf{b}$  and a permutation  $\gamma$  uniquely determine an allocation. An auctioneer who is

using the random priority rule chooses a fixed priority rule  $\gamma$  by randomizing uniformly over  $\Gamma$ . However, the resulting priority rule is not told to the players before they make their bids. When the priority rule  $\gamma$  is told to the players before they make their bids, the tie breaking rule is called a *fixed priority rule*. Dealing with a fixed priority rule simplifies notations and proofs, and in most cases, and in particular in this paper, results that are obtained with this tie breaking rule are identical to the results obtained with the random priority rule. Therefore we will assume this tie breaking rule. In contrast, in Section 7 we discuss a non-standard approach for analyzing directly the first-arrival tie breaking rule.

Unless we say specifically otherwise we assume in this paper a fixed priority rule.

Without loss of generality we assume that the fixed priority rule is defined by the natural order,  $\tilde{\gamma} = (1, 2, \dots, n)$ . That is, bidder  $i$  has a higher priority than bidder  $k$  if and only if  $i < k$ . Given this fixed priority rule we can make the following definitions, which apply to all position auctions:

We denote by  $s(\mathbf{b}, i)$  the position player  $i$  is assigned to when the bid profile is  $\mathbf{b}$ . The allocation determined by  $\mathbf{b}$  is denoted by

$$\mathbf{s}(\mathbf{b}) = (s(\mathbf{b}, 1), s(\mathbf{b}, 2), \dots, s(\mathbf{b}, n)).$$

For every  $j \in K \cup \{-1, m+1\}$  we denote by  $\delta(\mathbf{b}, j)$  the set of players assigned to position  $j$ . Note that for  $j \in K$ ,  $\delta(\mathbf{b}, j)$  contains at most one player.

### 2.2 The payment schemes

Let  $\alpha$  be a click-through rate vector. Each position  $j \in K \cup \{-1, m+1\}$  is associated with a payment function,  $p_j^\alpha : \mathbf{B} \rightarrow \mathbb{R}_+$ , where  $p_j^\alpha(\mathbf{b})$  is the payment for position  $j$  when the bid profile is  $\mathbf{b}$ . Naturally we assume that  $p_{-1}^\alpha$  is identically zero. However, we also assume that  $p_{m+1}^\alpha$  is identically zero. Hence, a participant who is not assigned a real position pays nothing.

We call the vector of payment functions  $\mathbf{p}^\alpha = (p_j^\alpha)_{j \in K}$  the *position payment scheme*.

**Remark:** Whenever  $\alpha$  is fixed or its value is clear from the context we will allow ourselves to omit the superscript  $\alpha$  from the payment and other functions.

We deal with anonymous position payment schemes, i.e. the players' payments to the auctioneer are not influenced by their identities. This is modeled as follows: Let  $\mathbf{b} \in \mathbf{B} = B_1 \times B_2 \times \dots \times B_n$  be a bid profile. We denote by  $b_{(j)}$  the  $j^{\text{th}}$  highest bid in  $\mathbf{b}$ . For  $j > n$  we let  $b_{(j)} = 0$ . For example if  $\mathbf{b} = (3, 7, 3, 0, 2)$  then  $b_{(1)} = 7, b_{(2)} = 3, b_{(3)} = 3, b_{(4)} = 2, b_{(5)} = 0$ . We denote  $\mathbf{b}^* = (b_{(1)}, \dots, b_{(n)})$ . Anonymity is modeled by the requirement that for every two bid profiles  $\mathbf{b}, \mathbf{d} \in \mathbf{B}$ ,  $\mathbf{p}(\mathbf{b}) = \mathbf{p}(\mathbf{d})$  whenever  $\mathbf{b}^* = \mathbf{d}^*$ . That is, for every position  $j$  there exists a real-valued function  $\tilde{p}_j$  defined over all ordered vectors of bids such that for every  $\mathbf{b} \in \mathbf{B}$   $p_j(\mathbf{b}) = \tilde{p}_j(\mathbf{b}^*)$ .

We further assume that a player never pays more than his bid. That is  $p_j(\mathbf{b}) \leq b_{(j)}$  for every  $\mathbf{b} \in \mathbf{B}$  and for every  $j \in K$ .

It is convenient in certain cases to describe the payment functions indexed by the players. Let  $G$  be a position auction with a position payment scheme  $\mathbf{p}$ .

For every player  $i$  we denote

$$q_i(\mathbf{b}) = p_{s(\mathbf{b}, i)}(\mathbf{b}),$$

and

$$\mathbf{q}(\mathbf{b}) = (q_1(\mathbf{b}), q_2(\mathbf{b}), \dots, q_n(\mathbf{b})).$$

Note that the correspondence  $p \rightarrow q$  is one-to-one. We call  $\mathbf{q}$  the *player payment scheme*. All our assumptions about the position payment schemes can be transformed to analogous assumptions about the player payment schemes. For convenience, a position auction will be described either by its position payment scheme or by its player payment scheme.

The utility function for player  $i$ ,  $w_i : V_i \times \mathbf{B} \rightarrow \mathbb{R}_+$  is defined as follows:

$$w_i(v_i, \mathbf{b}) = \alpha_{s(b,i)}(v_i - q_i(\mathbf{b})) = \alpha_{s(b,i)}(v_i - p_{s(b,i)}(\mathbf{b})).$$

### 2.3 Central position auctions

We next describe the payment schemes of three central position auctions.

**Self-price position auctions:** Each player who is assigned to a position with a positive click-through rate pays his own bid. That is, for every  $j \in K$  and every  $\mathbf{b} \in \mathbf{B}$

$$p_j(\mathbf{b}) = b_{(j)} \quad (1)$$

**Next-price position auctions:** In this auction (run with a slight variation by Google), every player who is assigned to a position with a positive click-through rate pays the bid of the player assigned to the position right after him if there is such a player, and zero otherwise. That is for every  $j \in K$  and for every  $\mathbf{b} \in B$

$$p_j(\mathbf{b}) = b_{(j+1)} \quad (2)$$

**VCG position auctions:** In a Vickrey-Clarke-Groves (VCG) position auction the payment function for position  $j \in K$  is defined as follows.<sup>6</sup> For every  $\mathbf{b} \in \mathbf{B}$

$$p_j^{vcg}(\mathbf{b}) = \frac{\sum_{k=j+1}^{m+1} b_{(k)}(\alpha_{k-1} - \alpha_k)}{\alpha_j} \quad (3)$$

Note that the VCG position auction is not the next-price position auction unless there is only one position and  $\alpha_1 = 1$ .

### 2.4 Mediators for position auctions

We denote by  $G = G(\alpha, \mathbf{p})$  the position auction with the click-through rate vector  $\alpha$  and the payment scheme  $\mathbf{p}$ . Recall that the set of types of  $i$  is  $V_i = (0, \infty)$ . Let  $\mathbf{V} = V_1 \times V_2 \times \dots \times V_n$  be the set of profile of types, and for every  $S \subseteq N$  let  $\mathbf{V}_S = \times_{i \in S} V_i$ .

A *mediator* for  $G$  is a vector of functions  $\mathbf{m} = (\mathbf{m}_S)_{S \subseteq N}$ , where  $\mathbf{m}_S : \mathbf{V}_S \rightarrow \mathbf{B}_S$ . The mediator  $\mathbf{m}$  generates a pre-Bayesian game  $G_{\mathbf{m}}$ , which is called the *mediated game*. In this game every player  $i$  receives his type  $v_i$  and can either send a type,  $\hat{v}_i$  (not necessarily the true type) to the mediator, or send a bid directly to the auction. If  $S$  is the set of players that send a type to the mediator, the mediator bids on their behalf  $\mathbf{m}_S(\hat{\mathbf{v}}_S)$ . Hence, the action set of player  $i$  in the mediated game is  $B_i \cup V_i$ , where conveniently  $V_i$  denotes both,  $(0, \infty)$ , and a copy of  $(0, \infty)$ , which is disjoint from

<sup>6</sup>We use the standard payment function of the VCG mechanism. A general VCG mechanism may be obtained from the standard one by adding an additional payment function to each player, which depends only on the types of the other players. Some authors (see e.g., [7]) call the standard VCG mechanism, the VC mechanism. According to this terminology we actually deal with VC position auctions. However, we decided to use the more common terminology.

$B_i$ . We introduce the following terminology: The *T-strategy* for a player in the mediated game is the strategy, in which this player uses the mediator's services and reports his true value to the mediator. The *T-strategy profile* is the profile of strategies in which every player is using the T-strategy. The T-strategy profile is an *ex post equilibrium* in the mediated game if for every player  $i$  and type  $v_i$ , and for every vector of types of the other players,  $\mathbf{v}_{-i}$ , the following two conditions hold:

**E1:**  $i$  is not better off when he gives the mediator the right of play and report a false type. That is, for every  $\hat{v}_i \in V_i$

$$w_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i})) \geq w_i(v_i, \mathbf{m}_N(\hat{v}_i, \mathbf{v}_{-i})).$$

**E2:**  $i$  is not better off when he bids directly. That is for every  $b_i \in B_i$ ,

$$w_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i})) \geq w_i(v_i, (b_i, \mathbf{m}_{N \setminus i}(\mathbf{v}_{-i}))).$$

Whenever the T-strategy profile is an ex post equilibrium in  $G_{\mathbf{m}}$ , the mediator  $\mathbf{m}$  implements an *outcome function* in  $G$ . This outcome function is denoted by  $\varphi^{\mathbf{m}}$ , and it is defined as follows:

$$\varphi^{\mathbf{m}}(\mathbf{v}) = (s(m_N(v)), q(m_N(v))).$$

Hence, the range of the function  $\varphi^{\mathbf{m}}$  is the Cartesian product of the set of allocations with  $\mathbb{R}_+^2$ .

## 3. IMPLEMENTING THE VCG OUTCOME FUNCTION BY MEDIATION

In general, except for the VCG position auction we do not expect position auctions to possess an ex post equilibrium.<sup>7</sup> Therefore, the behavior of the participants in most position auctions cannot be analytically predicted, and in practice it can form a non-efficient allocation: an allocation that does not maximize social surplus. In contrast, in the VCG position auction the truth-reporting strategy is a dominant strategy for every player, and the resulting allocation is efficient. Given a position auction  $G$  our goal is to construct a mediator that would implement the outcome function of the VCG position auction. This outcome function is defined as follows:

$$\varphi^{vcg}(\mathbf{v}) = (s(\mathbf{v}), q^{vcg}(\mathbf{v})).$$

**Definition:** Let  $G$  be a position auction. Let  $\mathbf{m}$  be a mediator for  $G$ . We say the  $\mathbf{m}$  implements the *VCG outcome function* in  $G$ , or that it implements  $\varphi^{vcg}$  in  $G$  if the T-strategy profile is an ex post equilibrium in  $G_{\mathbf{m}}$ , and  $\varphi^{\mathbf{m}} = \varphi^{vcg}$ .

We demonstrate our definitions so far by a simple example:

**EXAMPLE 1.** Consider a self-price auction  $G = G(\alpha, \mathbf{p})$  with 2 players and one position, with  $\alpha_1 = 1$ . That is,  $G$  is a standard two-person first-price auction. The corresponding VCG position auction is a standard second-price auction. We define a family of mediators  $\mathbf{m}^c$ ,  $c \geq 1$ , each of them implements the VCG position auction. Assume both

<sup>7</sup>Actually, it can be shown that if a strategy profile  $\mathbf{b}$  in a position auction is an ex post equilibrium then for every player  $i$   $b_i$  is a dominant strategy. It is commonly conjectured that except for some extremely artificial combinatorial auctions, the VCG combinatorial auctions are the only ones with dominant strategies (see [10]).

players use the mediator's services and send him the types  $\hat{v} = (\hat{v}_1, \hat{v}_2)$ , then all mediators act similarly and as follows: If  $\hat{v}_1 \geq \hat{v}_2$  the mediator makes the following bids on behalf of the players:  $b_1 = \hat{v}_2$ , and  $b_2 = 0$ . If  $\hat{v}_2 > \hat{v}_1$ , the mediator makes the bids  $b_1 = 0$ ,  $b_2 = \hat{v}_1$ . If only one player uses the mediator services, say player  $i$ , then mediator  $\mathbf{m}^c$  bids  $b_i = c\hat{v}_i$  on behalf of  $i$ . We claim that for every  $c > 1$ , the T-strategy profile is an ex post equilibrium in the mediated game generated by  $\mathbf{m}^c$ . Indeed, assume player 2 reports his type  $v_2$  to the mediator, and consider player 1.

If  $v_1 \geq v_2$  then by using the T-strategy player 1 receives the position and pays  $v_2$ . Hence, 1's utility is  $v_1 - v_2$ . If player 1 deviates by using the mediator's services and reporting  $\hat{v}_1 \geq v_2$  his utility is still  $v_1 - v_2$ . If he reports  $\hat{v}_1 < v_2$  his utility will be 0. If player 1 does not use the mediator, he should bid at least  $cv_2$  in order to get the positions, and therefore his utility cannot exceed  $v_1 - v_2$ .

If  $v_1 < v_2$ , then the T-strategy yields 0 to player 1, and any other strategy yields a non-positive utility.

Obviously each of the mediators  $\mathbf{m}^c$  implements the VCG outcome function. Note however, that the T-strategy is not a dominant strategy when  $c > 1$ ; e.g. if  $v_1 > v_2$  and player 2 bids directly  $v_2$  (without using the mediator services), then bidding directly  $v_1$  is better for player 1 than using the T-strategy: in the former case player 1's utility is 0 and in the latter case her utility is negative.

It is interesting to note that this simple example can not be extended to general self-price position auctions, as will be discussed in section 4.

While each of the mediators  $\mathbf{m}^c$  in Example 1 implements the VCG outcome function, the mediator with  $c = 1$  has a distinct characteristic: a player who uses the T-strategy cannot get a negative utility. In contrast, for every  $c > 1$ , if say player 2 does not use the mediator services, participates directly and bids less than  $cv_1$ , then the T-strategy yields a negative utility of  $(1 - c)v_1$  to player 1. This motivates our definition of valid mediators:

Let  $G$  be a position auction. A mediator for  $G$  is *valid*, if for every player, using the T-strategy guarantees a non-negative level of utility.

Formally, a mediator  $\mathbf{m}$  for  $G$ , is *valid* if for every subset  $S \subseteq N$  and every player  $i \in S$   $w_i(v_i, m_S(\mathbf{v}_S), \mathbf{b}_{-S}) \geq 0$  for every  $\mathbf{b}_{-S} \in \mathbf{B}_{-S}$  and every  $\mathbf{v}_S \in \mathbf{V}_S$ .

## 4. MEDIATORS IN NEXT-PRICE POSITION AUCTIONS

We now show that there exists a valid mediator, which implements the VCG outcome function in next-price position auctions. Although in the following section we prove a more general result, we present this result first, given the importance of next-price position auctions in the literature and in practice. Our proof makes use of the following technical lemma.

LEMMA 1. Let  $\mathbf{p}^{vcg}$  be the VCG payment scheme.

1.  $p_j^{vcg}(\mathbf{b}) \leq b_{(j+1)}$  for every  $j \in K$ .
2.  $p_j^{vcg}(\mathbf{b}) \geq p_{j+1}^{vcg}(\mathbf{b})$  for every  $j = 1, \dots, m-1$  and for every  $\mathbf{b} \in \hat{B}$ , where for every  $j$ , equality holds if and only if  $b_{(j+1)} = b_{(j+2)} = \dots = b_{(m+1)}$ .

The proof of Lemma 1 is given in the full version. We can now show:

THEOREM 2. Let  $G$  be a next-price position auction. There exists a valid mediator that implements  $\varphi^{vcg}$  in  $G$ .

Theorem 2 follows from a more general theorem given in the next section. However, we still provide a proof since a more simple and intuitive mediator is constructed for this case.

*Proof of Theorem 2.* We define a mediator  $\mathbf{m}$ , which will implement the VCG outcome function in  $G$ : For every  $\mathbf{v} \in V$  let  $\mathbf{m}_N(\mathbf{v}) = \mathbf{b}(\mathbf{v})$ , where  $\mathbf{b}(\mathbf{v})$  is defined as follows: For every player  $i$  such that  $2 \leq s(\mathbf{v}, i) \leq m$  let  $\mathbf{b}_i(\mathbf{v}) = p_{s(\mathbf{v}, i)-1}^{vcg}(\mathbf{v})$ .<sup>8</sup> For every  $i \in \delta(\mathbf{v}, m+1)$ ,  $\mathbf{b}_i(\mathbf{v}) = p_m^{vcg}(\mathbf{v})$ . Let  $\mathbf{b}_{\delta(\mathbf{v}, 1)}(\mathbf{v}) = 1 + \max_{\{i: s(\mathbf{v}, i) \geq 2\}} \mathbf{b}_i(\mathbf{v})$ .

For every  $S \subseteq N$  such that  $S \neq N$  and for every  $\mathbf{v}_S \in V_S$  let  $\mathbf{m}_S(\mathbf{v}) = \mathbf{v}_S$ . This completes the description of the mediator  $\mathbf{m}$ .

We show that  $\varphi^{\mathbf{m}}(\mathbf{v}) = \varphi^{vcg}(\mathbf{v})$  for every  $\mathbf{v} \in V$ . Let  $\mathbf{v} \in V$  be an arbitrary valuation vector.

We have to show that  $s(\mathbf{b}(\mathbf{v})) = s(\mathbf{v})$  and that  $q(\mathbf{b}(\mathbf{v})) = q^{vcg}(\mathbf{v})$ :

We begin by showing that  $s(\mathbf{b}(\mathbf{v})) = s(\mathbf{v})$ . It is sufficient to show that whenever  $1 \leq s(\mathbf{v}, i) < s(\mathbf{v}, l) \leq m+1$  for some  $i \neq l$ , then  $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$ .

We first show it for  $s(\mathbf{v}, i) = 1$ , that is  $\delta(\mathbf{v}, 1) = i$ . In this case  $b_{\delta(\mathbf{v}, 1)}(\mathbf{v}) > b_j(\mathbf{v})$  for every  $j \neq i$ ,  $s(\mathbf{b}(\mathbf{v}), i) = 1$ . Therefore  $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$ . If  $s(\mathbf{v}, i) > 1$ , we distinguish between two cases.

1.  $v_i = v_l$ . Since  $s(\mathbf{v}, i) < s(\mathbf{v}, l)$ , the fixed priority rule implies that  $i < l$ . By the second part of Lemma 1,  $p_{s(\mathbf{v}, i)-1}^{vcg}(\mathbf{v}) \geq p_{s(\mathbf{v}, l)-1}^{vcg}(\mathbf{v})$ . Therefore  $\mathbf{b}_i(\mathbf{v}) \geq \mathbf{b}_l(\mathbf{v})$ , which yields  $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$ .
2.  $v_i > v_l$ . Let  $j+1 = s(\mathbf{v}, i)$ . That is  $v_{(j+1)} = v_i$ , and therefore by the second part of Lemma 1,  $p_{s(\mathbf{v}, i)-1}^{vcg}(\mathbf{v}) > p_{s(\mathbf{v}, i)}^{vcg}(\mathbf{v})$ . Since  $s(\mathbf{v}, i) \leq s(\mathbf{v}, l) - 1$ , by the second part of Lemma 1,  $p_{s(\mathbf{v}, i)}^{vcg}(\mathbf{v}) \geq p_{s(\mathbf{v}, l)-1}^{vcg}(\mathbf{v})$ . Therefore  $p_{s(\mathbf{v}, i)-1}^{vcg}(\mathbf{v}) > p_{s(\mathbf{v}, l)-1}^{vcg}(\mathbf{v})$ , which yields  $\mathbf{b}_i(\mathbf{v}) > \mathbf{b}_l(\mathbf{v})$ . Therefore  $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$ .

This completes the proof that  $s(\mathbf{b}(\mathbf{v})) = s(\mathbf{v})$  for all  $\mathbf{v} \in V$ .

Observe that for every player  $i$  such that  $s(\mathbf{b}(\mathbf{v}), i) \in K$

$$p_{s(\mathbf{b}(\mathbf{v}), i)}(\mathbf{b}(\mathbf{v})) = p_{s(\mathbf{v}, i)}^{vcg}(\mathbf{v}).$$

Therefore  $q_i(\mathbf{b}(\mathbf{v})) = q_i^{vcg}(\mathbf{v})$  for every  $i \in N$ . This shows that  $q(\mathbf{b}(\mathbf{v})) = q^{vcg}(\mathbf{v})$  for all  $\mathbf{v} \in V$ . Hence,  $\varphi^{\mathbf{m}} = \varphi^{vcg}$ .

We proceed to prove that the T-strategy is an ex-post equilibrium. Note that by the truthfulness of VCG, it is not beneficial for any player  $i$  to miss report her value to the mediator, given that all other players use the T-strategy. Next we show that it is not beneficial for a single player  $i \in N$  to participate in the auction directly if all other players use the T-strategy. Fix some  $\mathbf{v} \in V$ . Assume that player  $i$  is the only player that participates directly in the auction. Hence,  $\mathbf{v}_{-i}$  is the vector of bids submitted by the mediator. Let  $b_i$  be player  $i$ 's bid. Let  $k = s(\mathbf{v}, i)$ . Therefore, since  $\varphi^{\mathbf{m}} = \varphi^{vcg}$ ,  $s(\mathbf{b}(\mathbf{v}), i) = k$ . Let  $j$  be player  $i$ 's position in the deviation. Hence  $j = s((\mathbf{v}_{-i}, b_i), i)$ . If  $j \notin K$  then player  $i$ 's

<sup>8</sup>Recall that  $s(\mathbf{b}, i)$  denotes the position of player  $i$  under the bid profile  $\mathbf{b}$ , and  $\delta(\mathbf{b}, j)$  denotes the set of players assigned to position  $j$ . Whenever  $j \in K$ , we slightly abuse notations, and also refer to  $\delta(\mathbf{b}, j)$  as the player that is assigned to position  $j$ .

utility is zero and therefore deviating is not worthwhile for  $i$ . Suppose  $j \in K$ . Then

$$\begin{aligned}\alpha_k(v_i - p_k(\mathbf{b}(\mathbf{v}))) &= \alpha_k(v_i - p_k^{vcg}(\mathbf{v})) \geq \\ \alpha_j(v_i - p_j^{vcg}(\mathbf{v}_{-i}, b_i)) &\geq \alpha_j(v_i - v_{(j+1)}),\end{aligned}$$

where the first equality follows from  $\varphi^{\mathbf{m}} = \varphi^{vcg}$ , the first inequality follows since VCG is truthful, and the second inequality follows from the first part of Lemma 1. Since  $p_j$  is position  $j$ 's payment function in the next-price position auction,  $\alpha_j(v_i - v_{(j+1)}) = \alpha_j(v_i - p_j(\mathbf{v}_{-i}, b_i))$ . Therefore

$$\alpha_k(v_i - p_k(\mathbf{b}(\mathbf{v}))) \geq \alpha_j(v_i - p_j(\mathbf{v}_{-i}, b_i)).$$

Hence, player  $i$  does not gain from participating directly in the auction.

Finally we show that  $\mathbf{m}$  is valid. If all players choose the mediator then by the first part of Lemma 1 each player which uses the T-strategy will not pay more than his value. Consider the situation in which a subset of players,  $S$ , participate directly in the auction. Since the mediator submits the reported values on behalf of the other players, these other players will not pay more than their reported values. Hence a player which used the T-strategy will not pay more than his value.  $\square$

## 5. MEDIATORS IN GENERALIZED NEXT-PRICE POSITION AUCTIONS

In the previous section we discussed the implementation of the VCG outcome function in the next price position auction. In this section we deal with a more general family of position auctions, in which the payment of each player who has been assigned a position, is a function of the bids of players assigned to "lower" positions than his own. The payment scheme  $\mathbf{p}$  of such a position auction satisfies the following condition:

**N1:** For every  $j \in K$  and every  $\mathbf{b}^1, \mathbf{b}^2 \in B$  such that  $b_{(l)}^1 = b_{(l)}^2$  for every  $l > j$ , we have that  $p_j(\mathbf{b}^1) = p_j(\mathbf{b}^2)$ .

We next provide sufficient conditions for implementing the VCG outcome function by a valid mediator in position auctions whose payment schemes satisfy N1.

We need the following notation and definition. For every position auction  $G$  and every  $\mathbf{b} \in B$  let  $\varphi^G(\mathbf{b}) = (s(\mathbf{b}), q(\mathbf{b}))$ .

We say that  $G$  is a VCG cover if for every  $\mathbf{v} \in V$  there exists  $\mathbf{b} \in B$  such that  $\varphi^G(\mathbf{b}) = \varphi^{vcg}(\mathbf{v})$ .

We say that  $G$  is monotone if  $p_j(\mathbf{b}) \geq p_j(\mathbf{b}')$  for every  $j \in K$  and for every  $\mathbf{b} \geq \mathbf{b}'$ , where  $\mathbf{b} \geq \mathbf{b}'$  if and only if  $b_i \geq b'_i$  for every  $i \in N$ .

We are now able to show:

**THEOREM 3.** *Let  $G = G(\alpha, \mathbf{p})$  be a position auction such that  $\mathbf{p}$  satisfies N1. If the following conditions hold then there exists a valid mediator that implements  $\varphi^{vcg}$  in  $G$ :*

1.  $G$  is a VCG cover
2.  $G$  is monotone.

The proof of Theorem 3 is given in the full version. We next provide the construction of the valid mediator, which will implement the VCG outcome function in a position auction  $G$ , which satisfies the conditions of Theorem 3:

**Algorithm for building  $\mathbf{m}$  for  $G$ :**

- For every  $\mathbf{v} \in V$  let  $\mathbf{m}_N(\mathbf{v}) = \mathbf{b}(\mathbf{v})$ , where  $\mathbf{b}(\mathbf{v})$  is some bid profile such that  $\varphi^G(\mathbf{b}(\mathbf{v})) = \varphi^{vcg}(\mathbf{v})$

- For every  $i$  and for every  $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$ , let  $\mathbf{v}^i = (\mathbf{v}_{-i}, M(\mathbf{v}_{-i}))$ , where  $M(\mathbf{v}_{-i}) = 1 + \max_{j \neq i} v_j$ .
- For every  $i \in N$  and every  $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$ , let  $\mathbf{m}_{N \setminus \{i\}}(\mathbf{v}_{-i}) = \mathbf{b}_{-i}(\mathbf{v}^i)$ , where  $\mathbf{b}(\mathbf{v}^i)$  is some bid profile such that  $\varphi^G(\mathbf{b}(\mathbf{v}^i)) = \varphi^{vcg}(\mathbf{v}^i)$ .
- For every  $S \subseteq N$ , such that  $1 \leq |S| \leq n - 2$ , let  $\mathbf{m}_S(\mathbf{v}_S) = \mathbf{v}_S$ .

Remark: As we wrote, Theorem 3 applies in particular to next-price position auctions discussed in Section 4. However, this Theorem applies to many other interesting position auctions as will be shown later. Moreover, the mediator constructed for this general case is different from the one in the proof of Theorem 2.

We now show that condition N1 as well as the requirement that  $G$  is a VCG cover, and the requirement that  $G$  is monotone are all necessary for establishing our result. It is easy to see that if  $G$  is not a VCG cover then Theorem 3 does not hold. The following example shows the necessity of the monotonicity condition.

**EXAMPLE 4.** *Let  $G = G(\alpha, \mathbf{p})$  be the following position auction. Let  $n = 4$ ,  $m = 3$ ,  $\alpha = (100, 10, 1)$ ,  $p_1(\mathbf{b}) = b_{(2)} - b_{(3)}$  and  $p_2(\mathbf{b}) = \frac{b_{(3)} + b_{(4)}}{2}$ , and  $p_3(\mathbf{b}) = b_{(4)}$ . Notice that  $G$  is not monotone. Observe that condition N1 is satisfied. In the full version we show that  $G$  is a VCG cover, and it is not possible to implement the VCG outcome function in  $G$  with a valid mediator.*

The next example shows that Theorem 3 does not hold, when condition N1 is not satisfied.

**EXAMPLE 5.** *Let  $G = G(\alpha, \mathbf{p})$  be the following position auction. Let  $N = \{1, 2, 3\}$ ,  $K = \{1, 2\}$  and  $\alpha = (2, 1)$ . Let  $p_1(\mathbf{b}) = \frac{b_{(1)}}{4}$  and  $p_2(\mathbf{b}) = b_{(2)}$ . It is immediate to see that the monotonicity condition is satisfied. We next show that  $G$  is a VCG cover. Let  $\mathbf{v} \in V$  be an arbitrary valuation vector. We need to find a bid profile  $\mathbf{b}(\mathbf{v})$  such that  $\varphi^G(\mathbf{b}(\mathbf{v})) = \varphi^{vcg}(\mathbf{v})$ . Note that  $p_1^{vcg}(\mathbf{v}) = \frac{v_{(2)} + v_{(3)}}{2}$  and  $p_2^{vcg}(\mathbf{v}) = v_{(3)}$ . We define the bid profile  $\mathbf{b}(\mathbf{v})$  as follows.*

*Let  $b_{\delta(\mathbf{v}, 3)}(\mathbf{v}) = \frac{v_{(3)}}{2}$ ,  $b_{\delta(\mathbf{v}, 2)}(\mathbf{v}) = v_{(3)}$  and  $b_{\delta(\mathbf{v}, 1)}(\mathbf{v}) = 2v_2 + 2v_{(3)}$ .*

*By the construction of  $\mathbf{b}(\mathbf{v})$ ,  $s(\mathbf{b}(\mathbf{v}), i) = s(\mathbf{v}, i)$  for  $i = 1, 2, 3$ . In addition observe that  $p_j(\mathbf{b}(\mathbf{v})) = p_j^{vcg}(\mathbf{v})$  for  $j = 1, 2, 3, 4$ . Therefore  $\varphi^G(\mathbf{b}(\mathbf{v})) = \varphi^{vcg}(\mathbf{v})$ . Since  $\mathbf{v}$  is arbitrary,  $G$  is a VCG cover.*

*Naturally N1 is not satisfied. Suppose in negation that there exists a valid mediator  $\mathbf{m}$ , which implements the VCG outcome function in  $G$ . Consider the following vector of valuations  $\mathbf{v} = (12, 10, 8)$ . If all players use the mediator then player 2 (with valuation 10) gets position 2, pays 8, and therefore her utility is  $1(10 - 8) = 2$ . Player 2 can always bid more than the other players, and by that cause some other player to be positioned second; Since the mediator is required to be valid it must be that the mediator submits not more than 12 on behalf of both players 1 and 3. But then player 2 can bid 13, and win the first position; therefore, player 2's utility will be  $2(10 - \frac{13}{4}) > 8$ . This contradicts that  $\mathbf{m}$  is a valid mediator that implements the VCG outcome function in  $G$ .*

To summarize, we have shown sufficient conditions for transforming a large class of position auctions to the VCG

position auction by mediation. Moreover by dropping any of our conditions we get that such transformation might not be feasible.

In the next subsections we provide classes of interesting position auctions which can be transformed to the VCG position auction by mediation. These auctions satisfy the conditions of Theorem 3. However, in order to use Theorem 3 one has to check that a certain position auction,  $G$  is a VCG cover. In the full version paper, before we apply this theorem we present another useful theorem that gives sufficient conditions guaranteeing that  $G$  is a VCG cover.

### 5.1 Generalized next-price position auctions

In a generalized next-price auction the payment scheme is defined as follows: For every  $j \in K$  and for every  $\mathbf{b} \in B$   $p_j(\mathbf{b}) = b_{l(j)}$  where  $l(j)$  is an integer such that  $l(j) > j$ .<sup>9</sup>

We show:

**PROPOSITION 1.** *Let  $G$  be a generalized next-price position auction. There exists a valid mediator that implements  $\varphi^{vcg}$  in  $G$  if and only if the following two conditions hold: (i)  $l(j+1) > l(j)$  for  $j = 1, \dots, m-1$ , and (ii)  $l(m) \leq n$ .*

### 5.2 $K$ -next-price position auctions

In  $k$ -next-price position auctions the payment scheme is defined as follows: For every  $j \in K$  and for every  $\mathbf{b}$   $p_j(\mathbf{b}) = b_{(j+k)}$ .  $K$ -next-price position auctions are, in particular generalized next-price position auctions. Therefore Proposition 1 yields as a corollary:

**PROPOSITION 2.** *Let  $k \geq 1$ . Let  $G$  be a  $k$ -next-price position auction. There exists a valid mediator that implements  $\varphi^{vcg}$  in  $G$  if and only if  $n \geq m+k-1$ .*

### 5.3 Weighted next-price position auctions

In weighted next-price position auctions the payment schemes are of the following form. For every  $j \in K$  and for every  $\mathbf{b} \in B$ ,  $p_j(\mathbf{b}) = \frac{b_{(j+1)}}{c_j}$ , where  $c_j \geq 1$ .

**PROPOSITION 3.** *Let  $G$  be a weighted next-price position auction with the weights  $c_1, c_2, \dots, c_m$ . There exists a valid mediator that implements  $\varphi^{vcg}$  in  $G$  if and only if  $c_1 \geq \dots \geq c_m$ .*

### 5.4 Google-like position auctions

Google-like ad auctions are slightly different from next-price auction. In these auctions the click-through rate of an ad  $i$  in position  $j$  is the product of the "quality" of ad  $i$ ,  $\beta_i > 0$ , and the position click-through rate  $\alpha_j > 0$ .<sup>10</sup> Players are ranked in the positions by  $b_i\beta_i$ .

Let  $\mathbf{b} \in B$ . Let  $\tilde{\delta}(\mathbf{b}, j)$  be defined as follows. For every  $j \in K$ , let  $\tilde{\delta}(\mathbf{b}, j)$  be the player  $i$  that obtains position  $j$ , and for  $j = m$  let  $\tilde{\delta}(\mathbf{b}, j) = i$ , where  $i$  obtained position  $m$  in case there is more than one player  $i$  such that  $b_i = b_{(m)}$ , then  $i$  is chosen between them via the breaking rule  $\tilde{\gamma}$ .

If player  $i$  obtains position  $j \in K$  then she pays  $p_j(\mathbf{b}) = \frac{\beta_{\tilde{\delta}(\mathbf{b}, j+1)} b_{\tilde{\delta}(\mathbf{b}, j+1)}}{\beta_i}$ . Therefore player  $i$ 's utility will be

$$\alpha_j \beta_i \left( v_i - \frac{b_{\tilde{\delta}(\mathbf{b}, j+1)}}{\beta_i} \beta_{\tilde{\delta}(\mathbf{b}, j+1)} \right) = \alpha_j (v_i \beta_i - b_{\tilde{\delta}(\mathbf{b}, j+1)} \beta_{\tilde{\delta}(\mathbf{b}, j+1)}).$$

<sup>9</sup>Recall that  $b_{(j)} = 0$  for every  $j > n$

<sup>10</sup>See e.g. [17].

Hence by denoting  $\tilde{v}_i = v_i \beta_i$  for every  $i \in N$ , and by applying Theorem 2 we obtain:

**PROPOSITION 4.** *There exists a valid mediator which implements the VCG outcome function in the Google-like position auction.*

## 6. SELF-PRICE POSITION AUCTIONS

Let  $G$  be a self-price position auction as described at section 2. At example 1 we showed that when there is one position and two players, the VCG outcome function is implemented by a valid mediator in this auction. The proof in this example can be easily generalized to show that the VCG outcome function can be implemented by a valid mediator in a self-price position auction, in which there is one position and an arbitrary number of players,  $n \geq 2$ .

Next we show that it is impossible to implement the VCG outcome function, even by a non-valid mediator, in a self-price position auction which has more than one position ( $m > 1$ ).

**THEOREM 6.** *Let  $G$  be a self-price position auction with more than one position. There is no mediator that implements the VCG outcome function in  $G$ .*

*Proof.* Let  $\mathbf{v} \in V$  be the following valuation profile.  $v_n = 10$  and  $v_1 = v_2 = \dots = v_{n-1} = 5$ . The VCG outcome function assigns to this  $\mathbf{v}$  an allocation, in which player  $n$  receives position 1 and player 1 receives position 2. The payments of players  $n$  and 1 are both equal to 5. In order to implement such an outcome, a mediator must bid 5 on behalf of player  $n$  (so that this player pays 5), and it must bid less than 5 on behalf of any other player, because otherwise another player receives position 1. Note that the bid of any other player cannot equal 5 because every other player has an higher priority than  $n$ . In particular, even if player 1 gets indeed position 2 he will pay less than 5. Hence, no mediator can implement the VCG outcome function in  $G$ .□

The proof of Theorem 6 heavily uses the fixed priority rule assumption. However, as we have already said, all our results including this theorem hold also for the tie breaking rule defined by the random priority rule. The proof of the impossibility theorem for the random priority rule uses the fact that the particular bad priority rule used in the proof of Theorem 6, has a positive probability.

As we previously discussed, the fixed and random priority rules are just convenient ways to model the first-arrival rule, which is common in practice. When one attempts to directly model position auctions that use the first-arrival rule without these modeling choices he tackles a lot of modeling problems. In particular, it is not clear how to model a position auction with the first-arrival rule as a game with incomplete information. To do this, one has to allow a player not only to submit a bid but also to decide about the time of the bid. This raises a lot of additional modeling problems, such as determining the relationship between the time a player decides to submit a bid and the time in which this bid is actually recorded. Hence, efficient modeling as a game may be untractable. Nevertheless, in the next section we will analyze mediators in position auctions, which use the first-arrival rule. We will define ex post equilibrium and the notion of implementation by mediation without explicitly modeling well-defined games. We will show that in this case

there is a way to implement the VCG outcome function in a self-price position auction. Moreover, we will find a valid mediator that does the job.

## 7. POSITION AUCTIONS WITH THE FIRST ARRIVAL RULE

Let  $G$  be a position auction with the first-arrival rule. Every mediator for  $G$  has the ability to determine the order in which the bids he submits on behalf of the players are recorded; He can just submit the bids sequentially, waiting for a confirmation before submitting the next bid. We need the following notations.

Every order of bidding can be described by some  $\gamma \in \Gamma$ ;  $i$  bids before  $k$  if and only if  $\gamma_i < \gamma_k$ . Hence, an order of bids can serve as a priority rule. For every order of bids  $\gamma$  and a vector of bids  $\mathbf{b}$  we define  $s(\mathbf{b}, \gamma, i)$  as the position assigned to  $i$ . We denote the payment of  $i$  when the vector of bids is  $\mathbf{b}$  and the order of bidding is  $\gamma$  by  $q_i(\mathbf{b}, \gamma) = p_{s(\mathbf{b}, \gamma, i)}(b)$ , and we denote  $w_i(v_i, \mathbf{b}, \gamma)$  the utility of  $i$ .

A mediator for  $G$  should determine the bids of the players who use its services and also the order of bids as a function of the reported types. However, all mediators discussed in this paper will use the same rule to determine the order of bids: If all players report the vector of types  $\hat{\mathbf{v}}$ , the mediator uses the order of bids  $\gamma^{\hat{\mathbf{v}}}$ , which is defined as follows:  $\gamma_i^{\hat{\mathbf{v}}} < \gamma_k^{\hat{\mathbf{v}}}$  if and only if  $\hat{v}_i > \hat{v}_k$ , or  $\hat{v}_i = \hat{v}_k$  and  $i < k$ . For example, if  $n = 3$  and the reported types are  $\hat{\mathbf{v}} = (6, 7, 6)$ , then  $\gamma^{\hat{\mathbf{v}}} = (2, 1, 3)$ . If only a strict subset of the players use the mediator's services, the mediator applies the same order of bids rule to this subset. A mediator for a position auction with the first arrival rule is therefore defined by a vector  $\mathbf{m} = (\mathbf{m}_S)_{S \subseteq N}$ . However, such a mediator is called a *directed mediator* in order to stress the fact that it determines not only the bids but also the order of bids. To summarize: If all players use the directed mediator  $\mathbf{m}$ , and the reported bids are  $\hat{\mathbf{v}}$ , then the directed mediator bids  $\mathbf{m}_N(\hat{\mathbf{v}})_i$  on behalf of  $i$ ,  $i$  receives the position  $s(\hat{\mathbf{v}}, \gamma^{\hat{\mathbf{v}}}, i)$ , and pays  $q_i(\mathbf{m}_N(\hat{\mathbf{v}}), \gamma^{\hat{\mathbf{v}}})$ . If only the subset  $S$  uses the mediator's services, the reported types are  $\hat{\mathbf{v}}_S$ , and the other players bid directly  $\mathbf{b}_{-S}$  then the actual order of bids is not uniquely determined. If this order is  $\gamma$  then the position of  $i \in N$  is  $s(\mathbf{b}, \gamma, i)$ , and its payment is  $q_i(\mathbf{b}, \gamma)$ , where  $\mathbf{b} = (\mathbf{m}_S(\hat{\mathbf{v}}_S), \mathbf{b}_{-S})$ . In particular, if every player is using the T-strategy and the players' profile of types is  $\mathbf{v}$ , then the outcome generated by the directed mediator is

$$\psi^{\mathbf{m}}(\mathbf{v}) = (s(\mathbf{v}, \gamma^{\mathbf{v}}), \mathbf{q}(\mathbf{m}_N(\mathbf{v}), \gamma^{\mathbf{v}})).$$

But why should the players use the T strategy? Assume all players but  $i$  use the T strategy. If player  $i$  deviates from the T strategy by reporting a false type to the directed mediator, the resulting outcome is well-defined. On the other hand, when this player sends a bid directly to the auctioneer, the resulting outcome is not clear, because the order of bids is not clear.<sup>11</sup>

A good desired directed mediator would be one that no player would want to deviate from the T strategy independently of the order in which the bids are recorded because of his deviation. More specifically:

**Definition:** Let  $G$  be a position auction with the first-arrival rule, and let  $\mathbf{m}$  be a directed mediator for  $G$ . The

<sup>11</sup>It is clear however, that the resulting order  $\gamma$  is consistent with the well-defined order of bids of  $N \setminus i$ .

T-strategy profile is an *ex post equilibrium* with respect to  $\mathbf{m}$  if for every player  $i$  and type  $v_i$ , and for every vector of types of the other players,  $\mathbf{v}_{-i}$ , the following two conditions hold:

**F1:**  $i$  is not better off when he gives the directed mediator the right of play and reports a false type. That is, for every  $\hat{v}_i \in V_i$

$$w_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i}), \gamma^{(v_i, \mathbf{v}_{-i})}) \geq w_i(v_i, \mathbf{m}_N(\hat{v}_i, \mathbf{v}_{-i}), \gamma^{(\hat{v}_i, \mathbf{v}_{-i})}).$$

**F2:**  $i$  is not better off when he bids directly independently of the resulting order of recorded bids. That is for every  $b_i \in B_i$ , and for every  $\gamma \in \Gamma$ , which is consistent with the order of bids of members of  $N \setminus i$  resulting from the vector of types  $\mathbf{v}_{-i}$ ,

$$w_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i}), \gamma^{(v_i, \mathbf{v}_{-i})}) \geq w_i(v_i, (b_i, \mathbf{m}_{N \setminus i}(\mathbf{v}_{-i})), \gamma).$$

The notion of valid directed mediators is analogously defined:

**Definition:** Let  $G$  be a position auction with the first-arrival rule. A directed mediator for  $G$  is *valid*, if for every player, using the T-strategy guarantees a non-negative level of utility.

Formally, a directed mediator  $\mathbf{m}$  for  $G$  is *valid*, if for every player  $i$  with type  $v_i$ , for every subset  $S \subseteq N$  such that  $i \in S$ , for every  $\mathbf{v}_{S \setminus i}$ , and for every  $\mathbf{b}_{-S}$ ,  $w_i(v_i, \mathbf{m}_S(\mathbf{v}_S), \mathbf{b}_{-S}, \gamma) \geq 0$  for every  $\gamma \in \Gamma$ , which is consistent with the standard order of bids of  $S$  determined by the mediator when the reported types are  $\mathbf{v}_S$ .

The notion of implementation by mediation remains as before: The directed mediator  $\mathbf{m}$  implements the VCG outcome function in  $G$  if  $\psi^{\mathbf{m}} = \varphi^{vcg}$ .

Our previous results remain true for directed mediators for position auctions with the first arrival rule. Next we show that in contrast to Theorem 6, it is possible to implement the VCG outcome function in every self-price position auction with the first-arrival rule.

**THEOREM 7.** *Let  $G = G(\alpha, \mathbf{p})$  be the self-price position auction with the first arrival rule. There exists a valid directed mediator that implements the VCG outcome function in  $G$ .*

In the following theorem we provide sufficient conditions for implementing that the VCG outcome function in a position auction with the first-arrival rule. A special characteristic of auctions satisfying these sufficient conditions is that players' payments may depend also on their own bid, in contrast to the auctions discussed in Theorem 3. The long proof of this theorem is in the spirit of all previous proofs, and therefore it is omitted.

**THEOREM 8.** *Let  $G = G(\alpha, \mathbf{p})$  be a position auction with the first-arrival rule. If the following conditions hold then there exists a valid directed mediator for  $G$  that implements the VCG outcome function in  $G$ .*

1. For every  $\mathbf{v} \in V$  there exists  $\mathbf{b} \in B$  such that  $p_j(\mathbf{b}) = v_{(j)}$  for every  $j \in K$ .
2.  $G$  is monotone.
3.  $p_j(\mathbf{b}) \geq p_{j+1}(\mathbf{b})$  for every  $j \in K$  and every  $\mathbf{b} \in B$ .
4. For every  $j \in K$  and every  $\mathbf{b}^1, \mathbf{b}^2 \in B$  such that  $b_{(l)}^1 = b_{(l)}^2$  for every  $l \geq j$ ,  $p_j(\mathbf{b}^1) = p_j(\mathbf{b}^2)$ .



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