On the Value of Correlation

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Abstract. Correlated equilibrium [1] generalizes Nash equilibrium to allow correlation devices. Aumann showed an example of a game, and of a correlated equilibrium in this game, in which the agents' surplus (expected sum of payoffs) is greater than their surplus in all mixed-strategy equilibria. Following the idea initiated by the price of anarchy literature [2, 3] this suggests the study of two major measures for the value of correlation in a game with non-negative payoffs:

1. The ratio between the maximal surplus obtained in a correlated equilibrium to the maximal surplus obtained in a mixed-strategy equilibrium. We refer to this ratio as the mediation value.
2. The ratio between the maximal surplus to the maximal surplus obtained in a correlated equilibrium. We refer to this ratio as the enforcement value.

In this work we initiate the study of the mediation and enforcement values, providing several general results on the value of correlation as captured by these concepts. We also present a set of results for the more specialized case of congestion games [4], a class of games that received a lot of attention in the recent literature.

1 Introduction

One of the most famous and fruitful contributions to game theory has been the introduction of correlated equilibrium by Aumann (1974) . Consider a game in strategic form. A correlated strategy is a probability distribution over the set of strategy profiles, where a strategy profile is a vector of strategies, one for each player. A correlated strategy is utilized as follows: A strategy profile is selected according to the distribution, and every player is informed about her strategy in the profile. This selected strategy for the player is interpreted as a recommendation of play. Correlated strategies are most natural, since they capture the idea of a system administrator/reliable party who can recommend behavior but can not enforce it. Hence, correlated strategies make perfect sense in the context of congestion control, load balancing, trading, etc. A correlated strategy is called a correlated equilibrium if it is better off for every player to obey her recommended strategy if she believes that all other players obey their recommended strategies[1]. A major potential benefit of correlated equilibrium is to attempt to improve the social welfare of selfish players. In this paper, the social welfare obtained in a mixed-strategy profile is defined to be the expected sum of the payoffs of the players, and it is referred to as the surplus obtained in this profile.

A striking example introduced in Aumann's seminal paper [1] is of a two-player two-strategy game, where the surplus obtained in a correlated equilibrium is higher than the surplus obtained in every mixed-strategy equilibrium of the game. As a result, Aumann's example suggests that correlation may be a way to improve upon social welfare while still assuming that players are rational in the classical game-theoretic sense.[2]

A modification of Aumann's example serves us as a motivating example:
Aumann's Example

	b^1	b^2
a^1	5,1	0,0
a^2	4,4	1,5

In this game, there are three mixed-strategy equilibrium profiles. Two of them are obtained with pure strategies, (a^1, b^1), and (a^2, b^2). The surplus in each of these pure-strategy equilibrium profiles equals six. There is an additional mixed-strategy equilibrium in which, every players chooses each of her strategies with equal probabilities. The surplus obtained in this profile equals 5 $(= \frac{1}{4}(6+0+8+6))$ because every entry in the matrix is played with probability $\frac{1}{4}$. Hence, the maximal surplus in a mixed-strategy equilibrium equals 6. Consider the following correlated strategy: a probability of 1/3 is assigned to every pure strategy profile but (a^1, b^2). This correlated strategy is a correlated equilibrium. Indeed, when the row player is recommended to play a^1 she knows that the other player is recommended to play b^1, and therefore she strictly prefers to play a^1. When the row player is recommended to play a^2 the conditional probability of each of the columns is half, and therefore she weakly prefers to play a^2. Similar argument applied to the column player shows that the correlated strategy is indeed a correlated equilibrium. The surplus associated with this correlated equilibrium equals $\frac{20}{3}$ $(= \frac{1}{3}(6+8+6))$.

[1] Every correlated strategy defines a Bayesian game, in which the private signal of every player is her recommended strategy. It is a correlated equilibrium if obeying the recommended strategy by every player is a pure-strategy equilibrium in the Bayesian game.

[2] Other advantages are purely computational ones. As has been recently shown correlated equilibrium can be computed in polynomial time even for structured representations of games [5, 6].

The above discussion suggests one may wish to consider the value of correlation in games. In order to address the challenge of studying the value of correlation, we tackle two fundamental issues:

– How much can the society/system gain by adding a correlation device, where we assume that without such a device the agents play a mixed strategy equilibrium.
– How much does the society/system loose from the fact that the correlation device can only recommend (and can not enforce) a course of action?

We introduce two measures, namely the *mediation value* and the *enforcement value*. The mediation value will measure the ratio between the maximal surplus in a correlated equilibrium to the maximal surplus in a mixed-strategy equilibrium. Notice that the higher this number is, the more correlation helps. This concept relates to the price of anarchy[3] as follows: When translating the definition of price of anarchy to games with payoffs and not with costs,[4] the price of anarchy is the ratio between the maximal surplus to the minimal surplus obtained in a mixed-strategy equilibrium. The higher this number is, the value of a center is higher, where a center can enforce a course of play. Hence, the price of anarchy could have been called the value of a center with respect to anarchy, where a center can dictate a play, and when anarchy is measured by the worst social outcome reached by rational and selfish agents. The mediation value is the value of a center with respect to anarchy, where a center is reliable and can recommend a play, and anarchy is measured by the best social outcome reached by rational and selfish agents.[5]

In Aumman's example it can be shown that the correlated equilibrium introduced above is the best correlated equilibrium, i.e. it attains the maximal surplus among all correlated equilibria in the game. Hence, the mediation value of Aumann's game is $\frac{10}{9}$.

The enforcement value measures the ratio between the maximal surplus to the maximal surplus in a correlated equilibrium. That is, it is the value of a super-center with respect to a center who can just use correlated devices in equilibrium. As the maximal surplus in Aumann game is 8, the enforcement value in this game equals $\frac{6}{5}$.

In order for the above measures to make sense we consider games with non-negative payoffs.

In this paper we establish general and basic results concerning the measures defined above. We will consider the mediation (enforcement) value of classes of games, where the mediation (enforcement) value of a class of games is defined

[3] The concept of the price of anarchy has received much attention in the recent computer science literature. See e.g., [7–11].

[4] In many situations it is indeed natural to deal with non-negative costs rather than payoffs. Such models require special treatment.

[5] Anarchy means playing in a mixed-strategy equilibrium. The phenomenon of multiple equilibria forces a modelling choice. Currently the choice between the best and worst social outcomes is a matter of taste.

as the greatest lower bound of the mediation (enforcement) values of the games in the class.

We start by considering general games. Aumann's example implies that the mediation value of the class of two-player two-strategy (2×2) games is at least $10/9$. We first show that the mediation value of this class is $4/3$. Hence, the mediation value in any 2×2 game is bounded from above by $4/3$, and this upper bound is tight. Next we move to show the power of correlation in more complex games. In order to do so we consider the two possible minimal extensions of 2×2 games: Two-player games with three strategies for one of the players and two strategies for the other, and three-player games with two strategies for all players. We show that the mediation values of the games in each of these classes of games are unbounded. That is, the mediation value equals ∞. This implies that the mediation value is ∞ for the class of games in which, at least one agent has three strategies and for the class of games with at least three players. Again, this should be interpreted as a positive result, showing the extreme power of correlation.

Considering the enforcement value, we first show that it equals ∞ for the class of 2×2 games. Moreover, in a setup with three players we show that the enforcement value of the class of three-player games without dominated strategies equals ∞.

Following these general results, we consider the important class of congestion games [4, 12]. Indeed, this class of games is perhaps the most applicable to the game theory and CS synergy. In particular, results regarding the price of anarchy have been obtained for congestion games. We restrict our discussion to simple congestion games. In a simple congestion game there is a set of facilities. Every facility j is associated with a payoff function w_j. Every player chooses a facility, say facility j, and receives $w_j(k)$, where k is the number of players that chose facility j.

For completeness we first deal with the simple case where we have only two players. In this case we show that if the players can choose among only two facilities then the mediation value is bounded from above by $4/3$, and that this bound is achieved. In the more general case, where there are m facilities, the mediation value is bounded by 2. However, if we consider facilities with non-increasing payoffs (i.e. a player's payoff is monotonically non-increasing in the number of agents using its selected facility) then the mediation value is 1.

We then move to the more general case of simple congestion games, where there are $n \geq 2$ players. We show that for the case of three players, even if there are two facilities with non-increasing payoffs then the mediation value is unbounded. However, if we have n players and the two facilities have non-increasing *linear* payoff functions then the mediation value is bounded from above by $\frac{\sqrt{5}+1}{2}$. On the other hand, we show that the mediation value can be higher than 1 considering even two symmetric (identical) facilities with non-increasing payoffs. This further illustrates the power of correlation. Nevertheless, we also show that if we have m symmetric facilities, where the facility payoff functions obey a concavity requirement, the best mixed-strategy equilibrium obtains the maximal surplus, and therefore both the mediation value and the enforcement value are 1.

Finally, we study the enforcement value in the natural case where we have n players who choose among m symmetric facilities (where the payoff function associated with a facility may be arbitrary). We give a general characterization of the cases where the enforcement value is 1, and as a result determine the situations where correlation allows obtaining maximal surplus. Few other results about the enforcement value are obtained as well.

Many of our proofs are omitted from this paper due to lack of space. The proofs rely mainly on duality theorems in linear programming. In order to illustrate these techniques we added a short appendix with a discussion of these techniques and a short sketch of proof of one of our theorems.

2 Preliminaries

A finite game $\Gamma = (N, (S^i)_{i\in N}, (u^i)_{i\in N})$ in strategic form is defined as follows. Let N be a nonempty finite set of players. For each $i \in N$, let S^i be a finite set of strategies of player i. Let $S = S^1 \times S^2 \times \cdots \times S^n$ be the set of strategy profiles (n-tuples). An element of S is $s = (s^i)_{i\in N}$. For each $i \in N$ and $s \in S$ let $s^{-i} = (s^1, ..., s^{i-1}, s^{i+1}, ...s^n)$ denote the strategies played by everyone but i. Thus $s = (s^{-i}, s^i)$. For each player $i \in N$, let $u^i : S \to \mathbf{R}$ be the payoff function of player i. $u^i(s)$ is the payoff of player i when the profile of strategies s is played. Γ is called a *nonnegative game* if all payoffs to all players are nonnegative, i.e $u^i : S \to \mathbf{R}_+$.

A player can also randomize among her strategies by using a mixed strategy - a distribution over her set of strategies. For any finite set C, $\Delta(C)$ denotes the set of probability distributions over C. Thus $P^i = \Delta(S^i)$ is the set of mixed strategies of player i. For every $p^i \in P^i$ and every $s^i \in S^i$, $p^i(s^i)$ is the probability that player i plays strategy s^i. Every strategy $s^i \in S^i$ is, with the natural identification, a mixed strategy $p_{s^i} \in P^i$ in which

$$p_{s^i}(t^i) = \begin{cases} 1 & t^i = s^i \\ 0 & t^i \neq s^i. \end{cases}$$

p_{s^i} is called a *pure* strategy, and s^i is interchangeably called a strategy and a pure strategy (when it is identified with p_{s^i}). Let $P = P^1 \times P^2 \times \cdots \times P^n$ be the set of mixed strategy profiles.

Unless otherwise specified we will assume that $N = \{1, 2, ..., n\}, \quad n \geq 1$.

Any $\mu \in \Delta(S)$ is called a *correlated strategy*. Every mixed strategy profile $p \in P$ can be interpreted as a correlated strategy μ_p in the following way. For every strategy profile $s \in S$ let $\mu_p(s) = \prod_{i=1}^n p^i(s^i)$. With slightly abuse of notation, for every $\mu \in \Delta(S)$, we denote by $u^i(\mu)$ the expected payoff of player i when the correlated strategy $\mu \in \Delta(S)$ is played, that is:

$$u^i(\mu) = \sum_{s\in S} u^i(s)\mu(s). \tag{1}$$

Whenever necessary we identify p with μ_p. Naturally, for every $p \in P$ let $u^i(p) = u^i(\mu_p)$. Hence $u^i(p)$ is the expected payoff of player i when the mixed strategy p is played.

We say that $p \in P$ is a *mixed-strategy equilibrium* if $u^i(p^{-i}, p^i) \geq u^i(p^{-i}, q^i)$ for every player $i \in N$ and for every $q^i \in P^i$.

Definition 1. *(Aumann 1974, 1987) A correlated strategy $\mu \in \Delta(S)$ is a correlated equilibrium of Γ if and only if for all $i \in N$ and all $s^i, t^i \in S^i$:*

$$\sum_{s^{-i} \in S^{-i}} \mu(s^{-i}, s^i)[u^i(s^{-i}, s^i) - u^i(s^{-i}, t^i)] \geq 0. \tag{2}$$

It is well-known and easily verified that every mixed-strategy equilibrium is a correlated equilibrium. Let $u(\mu) = \sum_{i=1}^n u^i(\mu)$. The value $u(\mu)$ is called the *surplus* at μ. Let $N(\Gamma)$ be the set of all mixed-strategy equilibria in Γ and let $C(\Gamma)$ be the set of all correlated equilibria in Γ. We define $v_C(\Gamma)$ and $v_N(\Gamma)$ as follows:

$$v_C(\Gamma) \triangleq \max\{u(\mu) : \mu \in C(\Gamma)\},$$

$$v_N(\Gamma) \triangleq \max\{u(p) : p \in N(\Gamma)\}.$$

Note that $v_N(\Gamma)$ and $v_C(\Gamma)$ are well defined due to the compactness of $N(\Gamma)$ and $C(\Gamma)$ respectively, and the continuity of u. Define $opt(\Gamma)$ (the maximal surplus) as follows:

$$opt(\Gamma) \triangleq \max\{u(\mu) : \mu \in \Delta(S)\} = max\{u(s) : s \in S\}.$$

The *mediation value* of a nonnegative game Γ is defined as follows:

$$MV(\Gamma) \triangleq \frac{v_C(\Gamma)}{v_N(\Gamma)}.$$

If both $v_N(\Gamma) = 0$ and $v_C(\Gamma) = 0$ we define $MV(\Gamma)$ to be 1. If $v_N(\Gamma) = 0$ and $v_C(\Gamma) > 0$ then $MV(\Gamma)$ is defined to be ∞. Denote by $EV(\Gamma)$ the *enforcement value* of a nonnegative game Γ. That is,

$$EV(\Gamma) \triangleq \frac{opt(\Gamma)}{v_C(\Gamma)}.$$

If both $v_C(\Gamma) = 0$ and $opt(\Gamma) = 0$ then we define $EV(\Gamma)$ to be 1. If $v_C(\Gamma) = 0$ and $opt(\Gamma) > 0$ then $EV(\Gamma)$ is defined to be ∞. Finally, for a class of games $\mathcal{C}$ we denote

$$MV(\mathcal{C}) \triangleq \sup_{\Gamma \in \mathcal{C}} MV(\Gamma); \quad \text{and} \quad EV(\mathcal{C}) \triangleq \sup_{\Gamma \in \mathcal{C}} EV(\Gamma).$$

We will also make use of the following notation and definitions. Let $\mathcal{G}$ be the class of all nonnegative games in strategic form. For $m_1, m_2, ..., m_n \geq 1$ denote by $\mathcal{G}_{m_1 \times m_2 \times \cdots \times m_n} \subseteq \mathcal{G}$ the class of all games with n players in which

$|S^i| = m_i$ for every player i. Let $s^i, t^i \in S^i$ be pure strategies of player i. We say that s^i *weakly dominates* (or just dominates) t^i, and t^i is *weakly dominated* (or dominated) by s^i if for all $s^{-i} \in S^{-i}$

$$u^i(s^i, s^{-i}) \geq u^i(t^i, s^{-i}),$$

where at least one inequality is strict. We say that s^i *strictly dominates* t^i, and t^i is *strictly dominated* by s^i if all of the above inequalities are strict. If $u^i(s^i, s^{-i}) = u(t^i, s^{-i})$ for all $s^{-i} \in S^{-i}$ then we will say that s^i and t^i are *equivalent* strategies for player i.

One of the tools we will need in order to prove some of our results is linear programming. For any game Γ in strategic form, $C(\Gamma)$ is exactly the set of feasible solutions for the following linear program $(\widehat{P})$. Moreover, $\mu \in C(\Gamma)$ is an optimal solution for $(\widehat{P})$ if and only if $u(\mu) = v_C(\Gamma)$.

$$\widehat{P} \quad \begin{aligned} & \max \sum_{s \in S} \mu(s) u(s) \\ & \text{s.t.} \\ & \mu(s) \geq 0 \quad \forall s \in S, \\ & \sum_{s \in S} \mu(s) = 1, \\ & \sum_{s^{-i} \in S^{-i}} \mu(s)[u^i(t^i, s^{-i}) - u^i(s)] \leq 0 \quad \forall i \in N, \forall s^i \in S^i, \forall t^i \in S^i, t^i \neq s^i. \end{aligned}$$

The dual problem has one decision variable for each constraint in the primal. We let $\alpha^i(t^i|s^i)$ denote the dual variable associated with the primal constraint:

$$\sum_{s^{-i} \in S^{-i}} \mu(s)[u^i(t^i, s^{-i}) - u^i(s)] \leq 0.$$

Let β denote the dual variable associated with the primal constraint $\sum_{s \in S} \mu(s) = 1$. Let $\alpha = (\alpha^i)_{i \in N}$ where $\alpha^i = (\alpha^i(t^i|s^i))_{t^i, s^i \in S^i}, \quad t^i \neq s^i$. The dual problem may be written:

$$\widehat{D} \quad \begin{aligned} & \min \beta \\ & \text{s.t.} \\ & \alpha^i(t^i|s^i) \geq 0 \quad \forall i \in N, \forall s^i \in S^i, \forall t^i \in S^i, t^i \neq s^i, \\ & \sum_{i \in N} \sum_{s^i \neq t^i \in S^i} \alpha^i(t^i|s^i)[u^i(t^i, s^{-i}) - u^i(s)] + \beta \geq u(s) \quad \forall s \in S. \end{aligned}$$

It is well known that problems $\widehat{P}$ and $\widehat{D}$ are feasible and bounded, and their objective values equal $v_C(\Gamma)$. The feasibility is a consequence of the existence of equilibrium (Nash (1951)).

The following three lemmas are needed for some of our results.

Lemma 1. *Let Γ be a game in strategic form. Let $s^i \in S^i$ be dominated by some other strategy $t^i \in S^i$. For any $\mu \in C(\Gamma)$, $\mu(s) = 0$ for all $s = (s^i, s^{-i})$ where $u^i(t^i, s^{-i}) > u^i(s^i, s^{-i})$.*

The proof follows directly from definition 1.

Lemma 2. *(Farkas [13]) Given a matrix A of dimensions $s \times t$ and a vector $b \in \mathbb{R}^s$, one and only one of the following systems has a solution:*

$$\begin{array}{rl} (i) & Ax \geq b, \qquad x \in \mathbb{R}^t \\ (ii) & y^T A = 0, \quad y^T b > 0 \quad y \in \mathbb{R}^s_+. \end{array}$$

The following is a variant of the Farkas lemma [13].

Lemma 3. *Given a matrix A with dimensions $s \times t$ and a vector $b \in \mathbb{R}^s$, one and only one of the following systems has a solution:*

$$\begin{array}{rl} (i) & Ax = b, \quad x \geq 0 \quad x \in \mathbb{R}^t \\ (ii) & \mu A >= 0, \quad \mu b < 0 \quad \mu \in \mathbb{R}^s. \end{array}$$

3 Results for General Games

We now deal with general games in strategic form.

3.1 The Mediation Value

In this section we show the overwhelming power of correlation in general games. However, we start with extending Aumann's result on the power of correlation in 2×2 games.

Two-person two-strategy games Aumann's example shows that a mediation value of $\frac{10}{9}$ can be obtained in a 2×2 game. In order to study the value of correlation we prove:

Theorem 1. $MV(\mathcal{G}_{2\times 2}) = \frac{4}{3}$.

For the proof we will need the following lemma [14]:

Lemma 4. *Let $\Gamma \in \mathcal{G}_{2\times 2}$. If there exist a correlated equilibrium in Γ which is not induced by a Nash equilibrium, then Γ has at least two pure strategy equilibria.*

Proof of theorem 1.
Let $\Gamma \in \mathcal{G}_{2\times 2}$ be the game in figure 1.

	b^1	b^2
a^1	a, b	j, k
a^2	m, n	c, d

Figure 1

By lemma (4) it is enough to look at games with two or more pure strategy equilibria. If there are four pure strategy equilibria then the mediation value is one. Let Γ possess exactly three pure strategy equilibria and w.l.o.g let (a^2, b^2) be the only strategy profile not in equilibrium. It must be that $c < j$ or $d < n$. Since (a^1, b^1) and (a^2, b^1) are in equilibrium then $m = a$. If $c < j$ then every correlated equilibrium $\mu \in C(\Gamma)$ satisfies $\mu(a^2, b^2) = 0$ by lemma 1. Hence $MV(\Gamma) = 1$. The proof is similar if $d < n$. Suppose there are exactly two pure strategy equilibria in Γ. Let these equilibria be in the same row. W.l.o.g let (a^1, b^1), (a^1, b^2) be in equilibrium. Therefore $b = k$. Observe that if player 1 plays strategy a^1 with probability one then any mixed strategy of player 2 will end in an equilibrium. Since there are exactly two pure strategy equilibria it must be that $m < a$ or $c < j$. If $m < a$ and $c < j$ then by lemma 1 every correlated equilibrium $\mu \in C(\Gamma)$ satisfies $\mu(a^2, b^1) = 0$ and $\mu(a^2, b^2) = 0$. Therefore $MV(\Gamma) = 1$. If $m = a$ then $c < j$. then every correlated equilibrium $\mu \in C(\Gamma)$ satisfies $\mu(a^2, b^2) = 0$ by lemma 1. Since (a^2, b^1) is not in equilibrium then $n < d$. We have $b = k$ and $n < d$ then again by lemma 1 every correlated equilibrium $\mu \in C(\Gamma)$ satisfies $\mu(a^2, b^1) = 0$. Therefore $MV(\Gamma) = 1$. We showed that if the two pure strategy equilibria are on the same row then the mediation value is one. Let the two equilibria be on a diagonal. W.l.o.g let (a^1, b^1) and (a^2, b^2) be in equilibrium. In [14] p.8 it is shown that if $c = j$ or $a = b$ or $b = k$ or $d = n$ then $C(\Gamma)$ is exactly the convex hull of $N(\Gamma)$. Hence there is no extreme point of $C(\Gamma)$ that is not a Nash equilibrium, and therefore the mediation value is one. Assume that $c \neq j$, $a \neq b$, $b \neq k$ and $d \neq n$. We will make the following assumptions:

1. $u(a^1, b^1) \leq u(a^2, b^2)$. In particular $a + b \leq c + d$.
2. $u(a^2, b^1) \geq u(a^2, b^2)$ - we are interested to maximize $MV(\Gamma)$, so either $u(a^1, b^2)$ or $u(a^2, b^1)$ should be greater than $u(a^2, b^2)$, otherwise $MV(\Gamma) = 1$. In particular $m + n \geq c + d$.

Let $\alpha = \frac{a-m}{c-j}$ and $\beta = \frac{b-k}{d-n}$ (Note that α and β are well defined since $c \neq j$ and $d \neq n$. $N(\Gamma) = \{((1,0),(0,1)),((0,1),(0,1)),((\frac{1}{1+\beta}, \frac{\beta}{1+\beta}),(\frac{1}{1+\alpha}, \frac{\alpha}{1+\alpha}))\}$.

Before continuing with the proof we will need the following geometric characterization of $C(\Gamma)$. In our case $C(\Gamma)$ is a polytope with five vertices $\mu_i \quad i = 1, ..., 5$ (see [15] or [14]). The following matrices $V_i \quad i = 1, 2, 3, 4, 5$ describe the five vertices (correlated equilibria) of the polytope where $V_i(j, k)$ is the probability that players 1 and 2 play their jth and kth strategies respectively in the correlated equilibrium μ_i.

$$V_1 = \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix}, V_2 = \begin{pmatrix} 0, 0 \\ 0, 1 \end{pmatrix}, V_3 = \begin{pmatrix} \frac{1}{(1+\alpha)(1+\beta)}, & \frac{\alpha}{(1+\alpha)(1+\beta)} \\ \frac{\beta}{(1+\alpha)(1+\beta)}, & \frac{\alpha\beta}{(1+\alpha)(1+\beta)} \end{pmatrix},$$

$$V_4 = \begin{pmatrix} \frac{1}{(1+\alpha+\alpha\beta)}, & \frac{\alpha}{(1+\alpha+\alpha\beta)} \\ 0, & \frac{\alpha\beta}{(1+\alpha+\alpha\beta} \end{pmatrix}, V_5 = \begin{pmatrix} \frac{1}{(1+\beta+\alpha\beta)}, & 0 \\ \frac{\beta}{(1+\beta+\alpha\beta)}, & \frac{\alpha\beta}{(1+\beta+\alpha\beta)} \end{pmatrix}.$$

Note that μ_1, μ_2 and μ_3 are (Nash) equilibria.

Continue of proof:
Clearly $m + n \leq 2(c + d)$.
The above assumptions lead to: $j + k \leq \frac{c+d+a+b}{2}$ which leaves to explore only the correlated equilibrium μ_5. By our assumptions $u(\mu_1) \leq u(\mu_2)$.

Case 1: $u(\mu_3) \geq u(\mu_2)$ (the social surplus in the mixed strategy equilibrium is higher than in the pure strategy equilibria).

$$u(\mu_3) \geq c + d \Rightarrow a + b + (j + k)\alpha + (m + n)\beta \geq (c + d)(1 + \alpha + \beta)$$

We obtain the following condition:

$$\alpha \leq \frac{\beta(m + n - c - d) + a + b - c - d}{c + d - j - k} \tag{3}$$

$$MV(\Gamma) = \frac{a + b + (m + n)\beta + (c + d)\alpha\beta}{1 + \beta + \alpha\beta} \times \frac{1 + \beta + \alpha + \alpha\beta}{a + b + (j + k)\alpha + (m + n)\beta + (c + d)\alpha\beta} \leq$$

$$\frac{1 + \beta + \alpha + \alpha\beta}{1 + \beta + \alpha\beta} \triangleq f_1(\alpha)$$

Note that α and β are positive. $f_1(\alpha)$ is therefore a non-decreasing function of α. Therefore, by (3), it is enough to look at the case where $a + b = c + d$. Set $K = \frac{m+n-c-d}{c+d-j-k}$.

$$m + n - c - d \leq a - c, \quad c + d - j - k \geq d - b.$$

Therefore by the assumption that $a + b \leq c + d$ we have that $K \leq 1$. Condition (3) becomes $\alpha \leq K\beta$ for $K \leq 1$.

$$f_1(\alpha) \leq \frac{1 + (K + 1)\beta + K\beta^2}{1 + \beta + K\beta^2} \leq \frac{1 + 2\beta + \beta^2}{1 + \beta + \beta^2} \leq \frac{4}{3} \tag{4}$$

where the second inequality follows from the fact that $f_1(\alpha)$ is non-decreasing in α and the last inequality follows from the fact that 1 maximizes $\frac{(\beta+1)^2}{(\beta+1)^2-\beta}$.

Case 2: $u(\mu_3) \leq u(\mu_2)$,
Condition (3) becomes

$$\alpha > \frac{\beta(m + n - c - d) + a + b - c - d}{c + d - j - k} \tag{5}$$

$$MV(\Gamma) = \frac{a + b + (m + n)\beta + (c + d)\alpha\beta}{(1 + \beta + \alpha\beta)(c + d)} \triangleq f_2(\alpha)$$

Note that $f_2(\alpha)$ is a non-increasing function of α. We distinguish between the following two cases.

1. $a+b=c+d$
   It is enough to look at the case where $j=k=0$.
$$f_2(\alpha) \leq \frac{c+d+(m+n)\beta+(m+n-c-d)\beta^2}{(1+\beta+\frac{(m+n-c-d)}{c+d}\beta^2)(c+d)}$$
   Set $c+d=x \Rightarrow m+n=tx$ where $1 \leq t \leq 2$. We now obtain:
$$f_2(\alpha) \leq \frac{x+tx\beta+(t-1)x\beta^2}{x+x\beta+(t-1)x\beta^2} = \frac{1+t\beta+(t-1)\beta^2}{1+\beta+(t-1)\beta^2} \leq \frac{4}{3} \tag{6}$$
   Where the last inequality is the exact same case as (4).
2. $a+b<c+d$
   Set $a+b=x$, then
$$c+d=tx, \quad m+n=kx \quad ,t>1, \quad 1 \leq k \leq 2 \tag{7}$$
$$kt = \frac{m+n}{a+b} \leq \frac{a+b+c+d}{a+b} = t+1 \Rightarrow t \leq k-1 \tag{8}$$
$$f_2(\alpha) \leq \frac{a+b+(m+n)\beta+(c+d)[\frac{\beta(m+n-c-d)+a+b-c-d}{c+d}]\beta}{(1+\beta+\frac{\beta(m+n-c-d)+a+b-c-d}{c+d})(c+d)} =$$
$$\frac{1+kt\beta+t(k-1)\beta^2+\beta-t\beta}{t+t\beta+t(k-1)\beta^2+\beta-t\beta} = \frac{1+\beta+t\beta(\beta+1)(k-1)}{t+t\beta^2(k-1)+\beta} \leq$$
$$\frac{1+\beta^2+2\beta}{1+\beta^2(k-1)+\beta} \leq \frac{(\beta+1)^2}{(\beta+1)^2-\beta} \leq \frac{4}{3}$$
   where the last 2 inequalities follow from (7) and (8)

We showed that the mediation value is bounded from above by $\frac{4}{3}$. It remains to show that this bound is tight.

We now show a family of games in which the mediation value approaches the above $\frac{4}{3}$ bound. Consider the family of games Γ_x shown in Figure 2 (a variant of Aumann's example) where $x>1$.

	b^1	b^2
a^1	x,1	0,0
a^2	x-1,x-1	1,x

Figure 2

In this game the pure strategy profiles (a^1, b^1) and (a^2, b^2) are in equilibrium and $u(a^1, b^1) = u(a^2, b^2) = x+1$. There is one more equilibrium in mixed strategies where each player assigns the probability 0.5 to each of her strategies, which yields a surplus lower than $x+1$. The correlated strategy $\mu \in \Delta(S)$ where each of the strategy profiles (a^1, b^1) ,(a^2, b^1) and (a^2, b^2) is played with equal probability 1/3 is in equilibrium and $u(\mu) = \frac{4x}{3}$. μ obtains the largest surplus among all correlated equilibria in the game (see the proof). Hence $MV(\Gamma_x) = \frac{4x}{3(x+1)}$. Therefore $MV(\Gamma_x) \to \frac{4}{3}$ when $x \to \infty$. $\Box$

General Games The above theorem shows that the mediation value of the class of 2×2 games is finite. A major question we face is whether such finite bound exists for more general classes of games. We now show that, perhaps surprisingly, the mediation value equals ∞ if we consider slightly more complex games. In particular, if we allow one of the players in a 2-player game to have at least three strategies, while the other remains with two strategies, then the mediation value already equals ∞. Similarly, if we allow three players each with two strategies then the mediation value, again equals ∞. Together, these results show the power of correlation when we move beyond 2×2 games.

Theorem 2. *$MV(\mathcal{G}_{m_1 \times m_2}) = \infty$ for every $m_1, m_2 \geq 2$ such that $\max(m_1, m_2) \geq 3$.*

Proof of theorem 2
It is enough to show that the result holds for $n = 2$ players, $m_1 = 3$ and $m_2 = 2$. Let $\Gamma_{x,\epsilon}$ be the following parametric $\mathcal{G}_{2\times 3}$ games (figure 3):

	b^1	b^2	b^3
a^1	$x, 1-\epsilon$	$z, 1$	$0, 0$
a^2	$0, z-\epsilon$	$z-1, z-1$	$1, z$

Figure 3

where $z > 2$ is fixed, $x > z$ and $0 < \epsilon < 0.5$.

$$N(\Gamma_{x,\epsilon}) = \{((1,0)(0,1,0)), ((0,1),(0,0,1)), ((\epsilon, 1-\epsilon), (\frac{1}{1+x}, 0, \frac{x}{1+x}))\}.$$

The surplus in the pure strategy equilibria $p \in N(\Gamma)$ is $u(p) = z + 1$. The surplus in the mixed strategies equilibrium is:

$$\frac{\epsilon(x+1-\epsilon) + (1-\epsilon)(z-\epsilon) + x(1-\epsilon)(z+1)}{x+1} =$$

$$\frac{z(x+1) - \epsilon z(x+1) + x}{x+1} \quad \to z+1$$

as $x \to \infty$ and $\epsilon \to 0$.

Let $\mu_i, \quad i = 1, ..., 6$ be the probabilities in a correlated equilibrium $\mu \in C(\Gamma)$. Let $V = \begin{pmatrix} \mu_1, \mu_2, \mu_3 \\ \mu_4, \mu_5, \mu_6 \end{pmatrix}$ where V(i,j) is the probability assigned to the strategy profile (a^i, b^j) in μ.
From the correlated equilibrium definition we obtain the following 9 equations:

1. $\mu_1 x + \mu_2 - \mu_3 \geq 0$.
2. $\mu_6 - \mu_5 - x\mu_4 \geq 0$.
3. $\mu_1 \leq \frac{(1-\epsilon)\mu_4}{\epsilon}$.
4. $\mu_1 \geq \frac{\epsilon\mu_4}{1-\epsilon}$.
5. $\mu_2 \geq \frac{(1-\epsilon)p_5}{\epsilon}$.

6. $\mu_2 \geq \mu_5$.
7. $\mu_6 \geq \frac{(1-\epsilon)\mu_3}{\epsilon}$.
8. $\mu_6 \geq \mu_3$.
9. $\sum_{i=1}^{6} \mu_i = 1$.

Set $\mu_1 = \epsilon, \mu_4 = 2\epsilon^2, \mu_2 = \epsilon(1-\epsilon), \mu_3 = \mu_5 = \epsilon^2, \mu_6 = 1 - \sum_{i=1}^{5} \mu_i$ and let $x = \frac{1}{4\epsilon^2}$. Let $\epsilon \to 0$. All equations are satisfied for every small enough ϵ. However $\lim_{\epsilon \to 0} x\mu_1 = \infty$.□

Theorem 3. *$MV(\mathcal{G}_{m_1 \times \cdots \times m_n}) = \infty$ for every $n \geq 3$ and for every $m_1, m_2, \cdot\cdot\cdot, m_n \geq 2$.*

Proof of theorem 3
We show this result for $n = 3$. Consider the following three player game Γ (figure 4):

c^1:

	b^1	b^2
a^1	0, 0, 0	2, 0, 0
a^2	0, 0, 0	1, 0, 1

c^2:

	b^1	b^2
a^1	4, 4, 0	0, 0, 1
a^2	5, 0, 0	0, 3, 0

Figure 4

We show that all equilibria have a zero surplus and we give a correlated equilibrium that has a strictly positive surplus. The only pure strategy equilibria in the game are $(a^1, b^1, c^1) \in S$ and $(a^2, b^1, c^1) \in S$. Every strategy profile where player 2 and 3 play b^1 and c^1 respectively (player 1 plays any mixed strategy) is in equilibrium. We next show that there are no more equilibria in the game. First we will see that there are no more equilibria where at least one of the players plays a pure strategy. We check this for each player:

1. Assume player 3 plays c^2 w.p. (with probability) one. If $1 > p^2(b^1) > 0$ then $p^1(a^2) = 1$, but then player 3 will deviate. If $p^2(b^1) = 1$ then $p^1(a^2) = 1$, but then player 2 will deviate. If $p^2(b^2) = 1$ then player 1 is indifferent. If $p^1(a^1) \geq 0.5$, player 2 will deviate. If $p^1(a^1) < 0.5$, player 3 will deviate. Assume player 3 plays c^1 w.p. one. If $p^2(b^2) > 0$ then $p^1(a^1) = 1$, but then player 3 will deviate. If $p^2(b^1) = 1$ then any convex combination of player 1 on her strategies results in an equilibrium, which it's surplus is zero.
2. Assume player 2 plays b^1 w.p. one. Player 3 is indifferent. If $p^3(c^2) > 0$ then $p^1(a^2) = 1$, but then player 2 will deviate. We saw the case $p^3(c^1) = 1$. Assume player 2 plays b^2 w.p. one. If $p^3(c^1) > 0$ then $p^1(a^1) = 1$, but then player 3 will deviate.
3. Assume player 1 plays a^1 w.p. one. If $p^2(b^2) > 0$ then $p^3(c^2) = 1$, but then player 2 will deviate. Assume player 1 plays a^2 w.p. one. If $p^3(c^2) > 0$ then $p^2(b^2) = 1$, but then player 3 will deviate.

We next show that there is no completely mixed equilibrium (an equilibrium where every player assigns a positive probability to all of her strategies). Let

$(p, 1-p)$, $(q, 1-q)$ and $(h, 1-h)$ be the strategies of player 1,2 and 3 respectively $(1 > p, q, h > 0)$. If these strategies are in equilibrium then the utility for player 2 is equal in columns b^1 and b^2. Hence $4p(1-h) = 3(1-p)(1-h)$ which implies that $p = \frac{3}{7}$. Similarly, the utility for player 3 is equal in matrices c^1 and c^2. Therefore $(1-p)(1-q) = p(1-q)$ yielding $p = 0.5$. I.e. there is no completely mixed equilibrium.
It remains to show that there exist a correlated strategy that has a strictly positive surplus. Let $\mu \in \Delta(S)$ be the following correlated strategy. $\mu(a^1, b^2, c^1) = \mu(a^2, b^2, c^1) = \mu(a^1, b^1, c^2) = \mu(a^2, b^1, c^2) = 0.25$ and for all other $s \in S$, $\mu(s) = 0$. We show that μ is a correlated equilibrium, which will imply that the mediation value is infinity since $u(\mu) > 0$. To see this observe the following inequalities (see 1) which define a correlated equilibrium:

1. $\mu(a^1, b^2, c^1)(2-1) + \mu(a^1, b^1, c^2)(4-5) \geq 0.$
2. $\mu(a^2, b^2, c^1)(1-2) + \mu(a^2, b^1, c^2)(5-4) \geq 0.$
3. $\mu(a^1, b^2, c^1)(0-0) + \mu(a^2, b^1, c^2)(0-0) \geq 0.$
4. $\mu(a^1, b^1, c^2)(4-0) + \mu(a^2, b^1, c^2)(0-3) \geq 0.$
5. $\mu(a^1, b^2, c^1)(0-1) + \mu(a^2, b^1, c^1)(1-0) \geq 0.$
6. $\mu(a^1, b^1, c^2)(0-0) + \mu(a^2, b^1, c^2)(0-0) \geq 0.$

□

3.2 The Enforcement Value

We first show that the enforcement value may be unbounded even on classes of small games.

Theorem 4. *$EV(\mathcal{G}_{m_1 \times \cdots \times m_n}) = \infty$ for every $n \geq 2$, and for every $m_1, m_2 \geq 2$.*

Proof of theorem 4
It is enough to prove for $n = 2$, $m_1 = 2$ and $m_2 = 2$. Consider the following (figure 5) parametric Prisoners' Dilemma games Γ_x, where $x > 1$:

	b^1	b^2
a^1	$1, 1$	$x+1, 0$
a^2	$0, x+1$	x, x

Figure 5

From lemma 1 every such game has a unique correlated equilibrium (a^1, b^1) where its surplus is 2. However for every $x \geq 1$ $opt(\Gamma_x) = 2x$. Therefore $EV(\Gamma_x) \to \infty$ whenever $x \to \infty$. □

The proof of Theorem 4 is shown using a parametric version of the well-known Prisoner's Dilemma game. This game has the property of possessing a strictly dominant strategy for each player. In the next theorem we show that dominance is not necessary for obtaining an unbounded enforcement value.

Theorem 5. $\sup\{EV(\Gamma) | \Gamma \in \mathcal{G}_{2\times2\times2}$,
no player has a strictly dominant strategy$\} = \infty$.

Proof of theorem 5
Consider the family of parametric games $\Gamma_{z,\epsilon}$ (Figure 6).

	a^1	a^2
a^1	$4-\epsilon, 4-\epsilon, 4-\epsilon$	$4, 4+\epsilon, 4$
a^2	$4+\epsilon, 4, 4$	$0, 0, z$

a^1

	a^1	a^2
a^1	$4, 4, 4+\epsilon$	$z, 0, 0$
a^2	$0, z, 0$	$0, 0, 0$

a^2

Figure 6

First observe that $opt(\Gamma_{F_\epsilon}) = z$ for every $0 < \epsilon \leq 0.25$. In order to prove the proposition result we use the dual program $(\widehat{D})$. Let (α, β) be a feasible solution for the dual problem then by the duality theorem $\beta \geq v_C(\Gamma)$. Let x_1, x_2, x_3 denote $\alpha^1(a_1|a_2)$, $\alpha^2(a_1|a_2)$ and $\alpha^3(a_1|a_2)$ respectively. Let y_1, y_2, y_3 denote $\alpha^1(a_2|a_1)$, $\alpha^2(a_2|a_1)$ and $\alpha^3(a_2|a_1)$ respectively. The dual constraints can be written in the following way (recall that $z = \frac{1}{\epsilon^2}$):

$$
\begin{aligned}
2\epsilon y_1 + 2\epsilon y_2 + 2\epsilon y_3 + \beta &\geq 12 - 3\epsilon,\\
-4y_1 - 4y_3 - 2\epsilon x_2 + \beta &\geq 12 + \epsilon,\\
-4y_2 - 4y_3 - 2\epsilon x_1 + \beta &\geq 12 + \epsilon,\\
-4y_1 - 4y_2 - 2\epsilon x_3 + \beta &\geq 12 + \epsilon,\\
-\frac{1}{\epsilon^2} y_1 + 4x_2 + 4x_3 + \beta &\geq \frac{1}{\epsilon^2},\\
-\frac{1}{\epsilon^2} y_2 + 4x_1 + 4x_3 + \beta &\geq \frac{1}{\epsilon^2},\\
-\frac{1}{\epsilon^2} y_3 + 4x_1 + 4x_2 + \beta &\geq \frac{1}{\epsilon^2},\\
\frac{1}{\epsilon^2} x_1 + \frac{1}{\epsilon^2} x_2 + \frac{1}{\epsilon^2} x_3 + \beta &\geq 0.
\end{aligned}
$$

Set $y_1 = y_2 = y_3 = x_1 = 0$, $\beta = \frac{1}{\epsilon}$, and $x_2 = x_3 = \frac{1}{4\epsilon^2}$, and observe that it is a feasible solution for every small enough ϵ. However, $\frac{z}{\beta} = \frac{1}{\epsilon} \to \infty$ as $\epsilon \to \infty$. $\square$

4 Simple Congestion Games

In this section we explore the mediation and enforcement values in simple congestion games. We first need a few notations and definitions.

A congestion form $F = (N, M, (X^i)_{i \in N}, (w_j)_{j \in M})$ is defined as follows. N is a nonempty set of players and M is a nonempty set of facilities. Unless otherwise specified we let $M = \{1, 2, ..., m\}$. For $i \in N$, let X^i be the set of strategies of player i, where each $A^i \in X^i$ is a nonempty subset of M. For $j \in M$ let $w_j \in R^{\{1,2,...,n\}}$ be the facility payoff function, where $w_j(k)$ denotes the payoff of each user of facility j, if there are exactly k users. A congestion form is nonnegative if for every $j \in M$ w_j is nonnegative. A congestion form is *simple* if for every $i \in N$, $X^i = \{\{1\}, \{2\}, ..., \{m\}\}$. Let $\mathcal{S}$ be the class of all nonnegative simple congestion forms and denote by $\mathcal{S}_{n\times m} \subseteq \mathcal{S}$ the class of all nonnegative simple congestion forms with n players and m facilities. Every congestion form $F = (N, M, (X^i)_{i\in N}, (w_j)_{j\in M})$ defines a *congestion game* $\Gamma_F = (N, (X^i)_{i\in N}, (u^i)_{i\in N})$ where N and X^i are as above and $(u^i)_{i\in N}$ is defined as follows. Let $X = \times_{i\in N} X^i$. For every $A = (A^1, A^2, ..., A^n) \in X$ and every $j \in M$ let $\sigma_j(A) = |\{i \in N : j \in A^i\}|$ be the number of users of facility j. Define $u^i : X \to \mathbb{R}$ by

$$u^i(A) = \sum_{j\in A^i} w_j(\sigma_j(A)). \tag{9}$$

Observe that if F is simple then $u^i(A) = w_{A^i}(\sigma_{A^i}(A))$.

We will say that a facility j is *non-increasing* if $w_j(k)$ is a non-increasing function of k. Define $\mathcal{SN}_{n\times m} \subseteq \mathcal{S}_{n\times m}$ as follows:

$\mathcal{SN}_{n\times m} \triangleq$
$\{F \in \mathcal{S}_{n\times m} |$all facilities in F are non-increasing$\}$.

We will call a facility j *linear* if there exist a constant d_j such that $w_j(k+1) - w_j(k) = d_j$ for every $k \leq 1$.

A congestion form is called *facility symmetric* or just *symmetric* if $w_j \equiv w_k \quad \forall j, k \in M$. Let $\mathcal{I}_{n\times m} \subseteq \mathcal{S}_{n\times m}$ be defined by

$$\mathcal{I}_{n\times m} \triangleq \{F \in \mathcal{S}_{n\times m} | \text{ F is facility symmetric } \}.$$

Define $\mathcal{IN}_{n\times m} \subseteq \mathcal{I}_{n\times m}$ as follows:

$\mathcal{IN}_{n\times m} \triangleq$
$\{F \in \mathcal{I}_{n\times m} |$ all facilities of F are non-increasing$\}$.

4.1 The Mediation Value

Although congestion games are especially interesting when the number of players is large, we first start with some results for the case where we have only two players, extending upon the results in the previous section. Following that, we will consider the more general n-player case.

The two-player case ($n = 2$) In theorem 1 we showed that $\frac{4}{3}$ is a tight upper bound for the mediation value of games that belong to $\mathcal{G}_{2\times 2}$. Hence, obviously, $\frac{4}{3}$ is an upper bound for the mediation value of simple congestion games with two players and two facilities, i.e. games generated by congestion forms in $\mathcal{S}_{2\times 2}$. We first show that this is also a tight upper bound.

Theorem 6. $MV(\{\Gamma_F | F \in \mathcal{S}_{2\times 2}\}) = \frac{4}{3}$.

Proof of theorem 6
Let F_x be the following form Let $M = \{a_1, a_2\}$ and let $w_{a_1} = (x, 0)$ and $w_{a_2} = (x-1, 1)$ where $x > 1$. The family of games Γ_{F_x} is the same family introduced in figure 1, only the columns exchanged places, which is similar to just rename the strategies of player 2. Therefore $MV(\Gamma_{F_x})$ remains the same. Hence $MV(\Gamma_{F_x}) \to \frac{4}{3}$ as $x \to \infty$. $\square$

Consider now the more general case where, the two agents can choose among m facilities. We show:

Theorem 7. $MV\left(\{\Gamma_F | F \in \mathcal{S}_{2\times m}\}\right) \leq 2$.

Proof of theorem 7
For every $j \in M$ let $w_j = (a_j, b_j)$ where $a_j, b_j \geq 0$. There exist at least one pure strategy equilibrium (by [4]). Let $j \in M$ and $k \in M$ be the facilities that player one and two choose respectively in a pure strategy equilibrium with the largest surplus. Denote by A this strategy profile.
Suppose $j \neq k$. Strategy profiles where each player chooses a different facility can't yield a higher surplus than $u(A)$ since A is in equilibrium. This implies that the only strategy profiles that can obtain a higher surplus than $u(A)$ are strategy profiles where both players choose the same facility l (l can be j or k). Let B be such a strategy profile. Suppose $w_l(2) > \max\{w_j(1), w_k(1)\}$. Since A is in equilibrium then $\min\{w_j(1), w_k(1)\} \geq w_f(1)$ for every $f \in M$. Therefore $w_l(2) \geq w_f(1)$ for every $f \in M$ which implies that B is in equilibrium and $u(B) > u(A)$ which contradicts that A is a pure strategy equilibrium that obtains the largest surplus. It must be that $w_l(2) \leq \max\{w_j(1), w_k(1)\}$ and therefore $u(B) \leq 2\max\{w_j(1), w_k(1)\} \leq 2u(A)$. We showed that if $j \neq k$ then $u(B)$ is no more than two times $u(A)$. Hence in the case where $j \neq k$ we showed that $MV(\Gamma_F) \leq 2$.
Suppose that $j = k$, i.e. both players choose facility j in A. Suppose $w_j(1) \leq w_j(2)$. Let C be a strategy profile where the players choose different facilities (one of them can choose j). Since A is in equilibrium then $w_l(1) \leq w_j(2)$ for every $l \neq j$. In addition we have $w_j(1) \leq w_j(2)$. Therefore $u(C) \leq u(A)$. Let D be a strategy profile where both players choose the same facility $h \neq j$. If $w_h(2) \leq w_j(2)$ then $u(D) \leq u(A)$. Otherwise $w_h(2) > w_j(2) \geq w_l(1)$ for every $l \in M$. But then D is in equilibrium and $u(D) > u(A)$ which contradicts the maximality of A.
Suppose $w_j(1) > w_j(2)$. Since A is in equilibrium $w_l(1) \leq w_j(2)$ for every $l \neq j$. First we will see that it must be that $w_l(1) < w_j(2)$ for every $l \neq j$. Suppose $w_h(1) = w_j(2)$ for some $h \neq j$. Therefore $w_j(1) > w_j(2) = w_h(1) \geq w_l(1)$ for every $l \neq j$. Therefore, if $w_j(1) \geq w_h(2)$ then the strategy profile where one player chooses h and the other chooses j is in equilibrium and obtains a larger surplus than A - a contradiction to the maximality of A. Otherwise if $w_h(2) > w_j(1)$ then since $w_j(1) > w_j(2) \geq w_l(1)$ for every $l \neq j$, the strategy profile where both players choose h is in equilibrium and obtains a larger surplus than A, again a contradiction to the maximality of A. Therefore $w_l(1) < w_j(2)$. If $w_l(2) \geq w_j(1)$ for some $l \neq j$ then the strategy profile where both players choose l would be in equilibrium and $2w_l(2) \geq 2w_j(1) > 2w_j(2)$ contradicting the maximality of A. Thus $w_l(2) < w_j(1)$ for every $l \neq j$. We showed that $w_j(2) > w_l(1)$ and $w_j(1) > w_l(2)$ for every $l \neq j$. In addition we assumed that $w_j(1) > w_j(2)$ and therefore $w_j(1) > w_l(1)$ for every $l \neq j$. Thus choosing facility j is a strictly dominant strategy. Therefore, by lemma 1, there exist a unique correlated equilibrium, i.e $MV(\Gamma_F) = 1$. $\square$

Notice that our results imply that correlation helps already when we have congestion games with only two players. However, correlation does not increase social welfare when all facility payoff functions are non-increasing:

Theorem 8. *$MV(\Gamma_F) = 1$ for every form $F \in \mathcal{SN}_{2\times m}$.*

Proof of theorem 8
For every $j \in M$ let $w_j = (a_j, b_j)$ where $a_j \geq b_j \geq 0$. Choose the largest two numbers between $a_1, b_1, ..., a_m, b_m$ while preferring a_j on b_k if $a_j = b_k$ for every $j, k \in M$. If the two largest chosen are a_j and a_k for some $j \neq k$ then the strategy profile where one player chooses facility j and the other chooses facility k is in equilibrium and this strategy profile also obtains the maximal surplus. Therefore $MV(\Gamma_F) = 1$. Otherwise for some $j \in M$, a_j and b_j are the largest. For every $k \neq j$ $w_j(1) \geq w_j(2) > w_k(1 \geq w_k(2)$. But this means that the strategy profile where both players choose facility j is in equilibrium and for both players choosing j is strictly dominant which leads also to a unique correlated equilibrium. Hence $MV(\Gamma_F) = 1$. $\square$.

Simple congestion games with n players In Section 3 we have shown that correlation has an unbounded value when considering arbitrary games. We next consider the effects of correlation in the context of simple congestion games. We can show:

Theorem 9. *$MV(\{\Gamma_F | F \in \mathcal{SN}_{n\times m}\}) = \infty$ for every $n \geq 3$ and for every $m \geq 2$.*

Proof of theorem 9
It is enough to prove the theorem for $n = 3$ and $m = 2$. Consider the following family of forms F_ϵ, $0 < \epsilon \leq 0.5$. Let $M = \{a_1, a_2\}$ and let $w_{a_1} = (z, 4, 4 - \epsilon)$, $w_{a_2} = (4 + \epsilon, 0, 0)$ where $z = \frac{1}{\epsilon^2}$. Notice that the monotonicity is satisfied for every $0 < \epsilon \leq 0.5$. Observe that the game F_ϵ is the game in the proof of theorem 5 (figure 6).

We will show that for every small enough ϵ the surplus in all equilibria is bounded from above by 13, while the surplus approaches infinity in a correlated equilibrium as $\epsilon \to 0$.

1. Pure strategy equilibrium. Any profile where two players play a^1 and one player plays a^2 is in equilibrium. The surplus in these strategy profiles is $12 + \epsilon$.
2. Mixed strategy equilibrium. Suppose one player plays strategy a^2 with probability one. In this case there is a unique equilibrium where the other two players play a^1 with probability one, i.e. a pure strategy equilibrium.
   If some player plays a^1 with probability one, the game between the other two players has the structure shown in figure 7.

	a^1	a^2
a^1	$4-\epsilon, 4-\epsilon$	$4, 4+\epsilon$
a^2	$4+\epsilon, 4$	$0, 0$

Figure 7

If for example player 1 plays a^1 w.p. one, then there exist a unique equilibrium which is:

$$((1,0),(\frac{2}{2+\epsilon},\frac{\epsilon}{2+\epsilon}),(\frac{2}{2+\epsilon},\frac{\epsilon}{2+\epsilon}))$$

From the above, every permutation of this vector is also an equilibrium. The surplus in all of these equilibria is

$$3(4-\epsilon)(\frac{2}{2+\epsilon})^2+2(12+\epsilon)\cdot\frac{2\epsilon}{(2+\epsilon)^2}+z\cdot(\frac{\epsilon}{2+\epsilon})^2=$$

$$\frac{48+36\epsilon+(z+4)\epsilon^2}{(2+\epsilon)^2}\quad\rightarrow 12.25$$

as $\epsilon\rightarrow 0$.

Assume all players assign positive probabilities to both of their strategies in an equilibrium. In particular let $((p,1-p),(q,1-q),(h,1-h))\quad 0<p,q,h<1$ be a completely mixed equilibrium. Therefore the utility for player 1 if she plays strategy a^1 is similar to her utility if she plays strategy a^2, i.e.

$$(4-\epsilon)qh+4(1-q)h+4(1-h)q+z(1-h)(1-q)=(4+\epsilon)qh. \tag{10}$$

Note that the same equations hold for players 2 and 3, where q and h are replaced by p respectively. Assume w.l.o.g. that h is fixed. Therefore given z and h, equation (10) has a unique solution q. Therefore $p=q$ (just replace q with p). Similarly if q is fixed then $p=h$. Thus, we obtain $p=q=h$. Equation (10) reduces to

$$(z-8-2\epsilon)p^2+(8-2z)p+z. \tag{11}$$

Solving (11) as $0<p<1$ we obtain

$$p=\frac{2z-8-\sqrt{64+8z\epsilon}}{2z-16-4\epsilon}. \tag{12}$$

The surplus in this equilibrium is:

$$3p(1-p)^2z+3p^2(1-p)(12+\epsilon)+p^3(12-3\epsilon)=$$

$$(3z-24-5\epsilon)p^3-(6z-36+3\epsilon)p^2+3zp. \tag{13}$$

Set $A=(24-5\epsilon)p^3-(36+3\epsilon)p^2$ and $B=3zp^3-6zp^2+3zp$. Hence (13) $=A+B$. We sill show that $A\rightarrow 12$ and $B\rightarrow 0$ as $\epsilon\rightarrow 0$. Observe that $p\rightarrow 1$ as $\epsilon\rightarrow 0$. This implies that $A\rightarrow 0$ as $\epsilon\rightarrow 0$. Observe that $z\epsilon=\sqrt{z}$. Thus, from (12) and for every small enough ϵ we have that $p\approx\hat{p}=\frac{2z-\sqrt{8}z^{\frac{1}{4}}}{2z}=1-\frac{\sqrt{8}}{2}z^{-\frac{3}{4}}$. For simplicity set $c=\frac{\sqrt{8}}{2}$.

$$B\approx 3z(1-cz^{-\frac{3}{4}})^3-6z(1-cz^{-\frac{3}{4}})^2+3z(1-cz^{-\frac{3}{4}})=$$

$$3z[1-3cz^{-\frac{3}{4}}+3c^2z^{-\frac{6}{4}}-c^3z^{-\frac{9}{4}}]-6z[1-2cz^{-\frac{3}{4}}+c^2z^{-\frac{6}{4}}]+3z(1-cz^{-\frac{3}{4}})=$$

$$3c^2z^{-\frac{2}{4}}-3c^3z^{-\frac{5}{4}}=$$

$$3c^2\epsilon-3c^3\epsilon^{\frac{10}{4}}\rightarrow 0$$

as $\epsilon\rightarrow 0$.

We showed that $v_N(\Gamma_{F_\epsilon}) \leq 13$ for every small enough ϵ.

It is enough to find a correlated equilibrium $\mu \in C(\Gamma_{F_\epsilon})$ such that $\lim_{\epsilon\to 0} u(\mu) = \infty$.

Let $\mu_i \quad i = 1, 2, ..., 8$ be the probabilities in a correlated equilibrium as follows:

$V_1 = \begin{pmatrix} \mu_1, \mu_2 \\ \mu_3, \mu_4 \end{pmatrix}$, $V_2 = \begin{pmatrix} \mu_5, \mu_6 \\ \mu_7, \mu_8 \end{pmatrix}$ where $V_k(i,j)$ is the probability that players 1,2 and 3 play their ith, jth and kth strategy respectively. From the definition of correlated equilibrium we obtain the following inequalities and equality:

1. $-2\epsilon\mu_1 + 4\mu_2 + 4\mu_5 + z\mu_6 \geq 0$.
2. $2\epsilon\mu_3 - 4\mu_4 - 4\mu_7 - z\mu_8 \geq 0$.
3. $-2\epsilon\mu_1 + 4\mu_3 + 4\mu_5 + z\mu_7 \geq 0$.
4. $2\epsilon\mu_2 - 4\mu_4 - 4\mu_6 - z\mu_8 \geq 0$.
5. $-2\epsilon\mu_1 + 4\mu_2 + 4\mu_3 + z\mu_4 \geq 0$.
6. $2\epsilon\mu_5 - 4\mu_6 - 4\mu_7 - z\mu_8 \geq 0$.
7. $\sum_{i=1}^{8} \mu_i = 1$.

Set $\mu_1 = \mu_7 = \mu_8 = 0$, $\mu_3 = \mu_5 = \epsilon^{\frac{1}{4}}$ $\mu_4 = \mu_6 = \epsilon^{\frac{3}{2}}$ and $\mu_2 = 1 - \sum_{i=3}^{6} \mu_i$.
It is easy to see that all the inequalities are satisfied for every small enough ϵ.
However $z\mu_4 = \frac{1}{\sqrt{\epsilon}} \to \infty$ as $\epsilon \to 0$. □

The above result illustrates the power of correlation when we consider the context of simple congestion games. Indeed, the result shows that even if there are three players and two facilities with non-increasing payoff functions the mediation value is unbounded. However, if we require that the facility payoff functions are linear, then the following upper bound can be obtained:

Theorem 10. *$sup\{MV(\Gamma_F)|F \in \mathcal{SN}_{n\times 2}$, all facilities of F are linear$\} \leq \phi$, where $\phi = (\sqrt{5}+1)/2$.*

Proof of theorem 10

Let $M = \{f, g\}$ and let w_f and w_g be the facility payoff functions of f and g respectively. W.l.o.g $w_f(1) \geq w_g(1)$. Let $d_f = w_f(k) - w_f(k+1)$ and $d_g = w_g(k) - w_g(k+1)$ for every $1 \leq k \leq n-1$. Let $\pi_k = (n-k, k)$ be the congestion vector where k players choose g and $n-k$ players choose f. Let s be the largest integer such that the congestion vector $\pi_s = (n-s, s)$ is in equilibrium (a pure strategy equilibrium exist due to Rosenthal [4]). The surplus in the above equilibrium is $u(\pi_s)$. If $s = n$ then $w_g(n) \geq w_f(1)$ and therefore $w_g(n) = w_f(1)$ which implies that $nw_g(n) = nw_f(1) \geq u(\pi_k)$ for every $n \geq k \geq 0$. Hence the strategy profile where all players choose g is in equilibrium and obtains the maximal surplus, i.e. the mediation value is one. If $s = 0$ then $w_f(n) > w_g(1)$ which implies that choosing f is a strictly dominant strategy and therefore there is a unique equilibrium. Therefore if $s = 0$ the mediation value is one. Consider the case where $1 \leq s \leq n-1$. We can assume by the above that $w_g(1) \geq w_f(n)$.

Claim 1: $u(\pi_j) \leq u(\pi_s)$ for every $j \leq s$.

Proof: For $j = s$ the result is trivial. Let $j = 0$. By the non-increasing property $w_f(n-s) \geq w_f(n)$, and since π_s is in equilibrium then $w_g(s) \geq w_f(n-s+$

$1) \geq w_f(n)$. Therefore $u(\pi_j) = nw_f(n) \leq sw_g(s) + (n-s)w_f(n-s) = u(\pi_s)$. Let $0 < j < s$. Note that by the linearity $w_g(j) = w_g(s) + (s-j)d_g$ and $w_f(n=j) = w_f(n-s) - (s-j)d_f$. Hence

$$u(\pi_s) - u(\pi_j) = sw_g(s) + (n-s)w_f(n-s) - j(w_g(s) + (s-j)d_g) - (n-j)(w_f(n-s) - (s-j)d_f) =$$

$$w_f(n-s)(j-s) + w_g(s)(s-j) - j(s-j)d_g + (n-j)(s-j)d_f =$$

$$(s-j)(w_g(s) - w_f(n-s) + (n-j)d_f - jd_g) \geq$$

$$(s-j)(w_f(n-s+1) - w_f(n-s) + (n-j)d_f - jd_g) \tag{14}$$

since π_s is in equilibrium. By the linearity of the form we have

$$(14) = (s-j)((n-j-1)d_f - jd_g) =$$

$$(s-j)(w_f(1) - w_f(n-j) - g(1) + g(j+1)) \geq 0.$$

Both $w_f(1) \geq w_g(1)$ and $w_g(j+1) \geq w_f(n-j)$ for every $1 \leq j \leq s+1$ yield the last inequality. □

Claim 2: For every $k > s$ every strategy profile in the following form is in equilibrium: $n-k$ players choose f with probability one, and the other k players choose g with probability $p_k = \frac{w_g(1) - w_f(n)}{(k-1)(d_f + d_g)}$. The surplus of such a strategy profile is $nw_f(n) + p_k d_f((n-k)k + k(k-1))$.

Proof: First we show that $p_k \leq 1$ for every $k > s$. It is enough to show this for $k = s+1$. Since π_s is in equilibrium we have $w_g(1) - w_f(n) \leq w_g(1) - w_g(s+1) + w_f(n-s) - w_f(n) = sd_g + sd_f$. Set $s = k-1$ to obtain that $p_k \leq 1$.

Let q_k be a strategy profile as described in the theorem for some $k > s$. Let $i \in N$ be a player that chooses facility f with probability one. For every player i $q_k^i = (r_f, r_g)$ is the strategy profile of player i where r_f and r_g are the probabilities that player i chooses facilities f and g respectively. We show that $u^i(q^{-i}, (1,0)) \geq u^i(q^{-i}, (0,1))$ for every $q^{-i} \in P^{-i}$. Recall that if $Z\ Bin(k,p)$ then $E(Z) = kp$.

$$u^i(q^{-i}, (1,0)) - u^i(q^{-i}, (0,1)) = \sum_{j=0}^{k} \binom{k}{j} p_k^j (1-p_k)^{k-j} (w_f(n-j) - w_g(j+1)) = \tag{15}$$

$$\sum_{j=0}^{k} \binom{k}{j} p_k^j (1-p_k)^{k-j} (w_f(1) - (n-j-1)d_f - w_g(1) + jd_g) \tag{16}$$

$$w_f(1) - w_g(1) - (n-1)d_f + kp_k d_f + kp_k d_g \geq \tag{17}$$

$$w_f(n) - w_g(1) + (k-1)p_k d_f + (k-1)p_k d_g \geq 0 \tag{18}$$

where the last inequality follows from $p_k = \frac{w_g(1) - w_f(n)}{(k-1)(d_f + d_g)} \leq 1$. We next show that for every player i that plays the mixed strategy $(1-p_k, p_k)$ is indifferent between f and g, i.e. $u^i(q^{-i}, (1,0)) = u^i(q^{-i}, (0,1))$.

$$u^i(q^{-i}, (1,0)) - u^i(q^{-i}, (0,1)) = \sum_{j=0}^{n} \binom{k-1}{j} p_k^j (1-p_k)^{k-1-j} (w_f(n-j) - w_g(j+1)) = \tag{19}$$

$$\sum_{j=0}^{n}\binom{k-1}{j}p_k^j(1-p_k)^{k-1-j}(w_f(1)-(n-j-1)d_f-w_g(1)+jd_g)= \tag{20}$$
$$w_f(1)-w_g(1)-(n-1)d_f+(k-1)p_kd_f+(k-1)p_kd_g= \tag{21}$$
$$w_f(n)-w_g(1)+(k-1)p_kd_f+(k-1)p_kd_g=0. \tag{22}$$

It remains to calculate the surplus at q_k.
By the above equations the payoff for each of the $n-k$ players that choose f w.p. one is $w_f(n)+kp_kd_f$, and the payoff for the other k players is $w_f(n)+(k-1)p_kd_f$. Therefore the surplus is $(n-k)(w_f(n)+kp_kd_f)+k(w_f(n)+(k-1)p_kd_f) = nw_f(n)+p_kd_f((n-k)k+k(k-1))$.□

Let q_{s+1} be a strategy profile such as in claim 2. In order to prove the theorem we use the dual program $\hat{D}$. Let (α,β) be a feasible solution for the dual problem, then by the duality theorem $\beta \geq v_C(\Gamma)$. Let $Z=\phi max\{u(\pi_s),u(q_{s+1})\}$. We show that there exist a feasible solution for the dual problem where $\beta \leq Z$.

Let $x=\alpha^i(f|g)$ and $\alpha^i(g|f)=0$ for every $i\in N$. The constraints of the dual program reduce to (call this system $\widehat{D_1}$):

$$\begin{array}{ll}\widehat{D_1} & k(w_f(n-k+1)-w_g(k))x \geq u(\pi_k)-\beta, \quad k=1,...,n\\ & x\geq 0.\end{array}$$

If $u(\pi_k)\leq Z$ for every k, then the result is immediate. Consider the case where there is at least one k such that $u(\pi_k)>Z$
Let $\hat{k}$ be such that $u(\pi_{\hat{k}})>Z$. From claim 1 it must be that $\hat{k}>s$. However $w_f(n-k+1)-w_g(k)>0$ for every $k>s$, otherwise $u(\pi_k)$ would be in equilibrium, which contradicts the selection of s. Hence, if there exists a feasible solution to $\widehat{D_1}$ where $\beta\leq Z$, it must be that $x\geq 0$. Therefore we can remove the constraint $x\geq 0$ from $\widehat{D_1}$ (the set of feasible solutions will not change under our assumptions). Call the new set of constraints (with out $x\geq 0$) $\widehat{D_2}$.
By Farkas lemma (2), $\widehat{D_2}$ has a solution if and only if the following program doesn't have a solution:

$$\begin{array}{ll}\widehat{P_1} & \sum_{k=1}^{n} y_k k(w_f(n-k+1)-w_g(k))=0\\ & \sum_{k=1}^{n} y_k(u(\pi_k)-\beta)>0\\ & y_k\geq 0 \quad k=1,...,n.\end{array}$$

At least one of the variables $y_k>0$ by the second constraint. Therefore we can assume w.l.o.g that $\sum_{k=1}^n y_k=1$, i.e. $y=(y_1,...,y_n)$ is a probability distribution. Let Y be the random variable where $y_k=P(Y=k)$ $\quad k=1,...,n$. Suppose that there exist a vector y that satisfies the first constraint. From the first constraint we have:

$$0=\sum_{k=1}^{n} y_k k(w_f(n-k+1)-w_g(k))=$$

$$\sum_{k=1}^{n} y_k k(w_f(n) + (k-1)d_f - w_g(1) + (k-1)d_g) =$$
$$EY(w_f(n) - w_g(1) - d_f - d_g) + E(Y^2)(d_f + d_g) \geq$$
$$EY(w_f(n) - w_g(1) - d_f - d_g) + (EY)^2(d_f + d_g).$$

Since $EY > 0$ we can divide both sides by EY and obtain

$$EY \leq \frac{w_g(1) - w_f(n) + d_f + d_g}{(d_f + d_g)} \leq s + 1 \tag{23}$$

where the last equality follows from the $(w_g(1) - w_f(n))/(d_f + d_g) \leq s$.

We proceed with the second constraint. Recall that $\sum_{k=1}^{n} y_k k w_f(n-k+1) = \sum_{k=1}^{n} k w_g(k)$ by the first constraint. It remains to show that $\sum_{k=1}^{n} y_k u(\pi_k) \leq \beta$ for any $\beta \leq Z$.

$$\sum_{k=1}^{n} y_k u(\pi_k) = \sum_{k=1}^{n} y_k(k w_g(k) + (n-k)w_f(n-k)) =$$
$$\sum_{k=1}^{n} y_k(k w_f(n-k+1) + (n-k)w_f(n-k)) =$$
$$\sum_{k=1}^{n} y_k k(w_f(1) - (n-k)d_f) + \sum_{k=1}^{n} y_k(n-k)w_f(n-k) =$$
$$w_f(1)EY + \sum_{k=1}^{n} y_k(n-k)(w_f(n-k) - kd_f) = w_f(1)EY + w_f(n)\sum_{k=1}^{n} y_k(n-k)$$
$$= EY(w_f(1) - w_f(n)) + nw_f(n). \tag{24}$$

We distinguish between two cases: (i) $p_{s+1} < 1/\phi$. (ii) $p_{s+1} \geq 1/\phi$.
(i) $p_{s+1} < 1/\phi$ implies that $(w_g(1) - w_f(n))/(d_f + d_g) \leq s/\phi$. ¿From (23) $EY \leq \frac{s}{\phi} + 1$. Let $\beta = Mu(\pi_s)$ where $M \geq 1$. Suppose that the second constraint is satisfied, i.e. (24) $= EY(w_f(1) - w_f(n) + nw_f(n) > Mu(\pi_s) - nw_f(n-s)$. Therefore

$$EY > \frac{Msw_g(s) + M(n-s)w_f(n-s) - nw_f(n)}{(n-1)d_f} =$$
$$\frac{Msw_g(s) + M(n-s)(w_f(n) + sd_f) - nw_f(n)}{(n-1)d_f} \geq$$
$$\frac{Msw_f(n-s+1) + (n(M-1) - Ms)w_f(n) + M(n-s)sd_f}{(n-1)d_f} \tag{25}$$

where the inequality follows from the fact that π_s is in equilibrium. Since $M \geq 1$ and $w_f(n) \geq 0$ we have

$$(25) \geq \frac{Ms(w_f(n-s+1) - w_f(n)) + M(n-s)sd_f}{(n-1)d_f} =$$

$$\frac{Ms(s-1)d_f + M(n-s)sd_f}{(n-1)d_f} = \frac{Ms(n-1)}{n-1} = Ms \tag{26}$$

Observe that $\frac{1}{\phi}+1=\phi$. Hence, every $M \geq \phi$ will contradict that $EY \leq \frac{s}{\phi}+1$.
(ii) Let $p_{s+1} \geq 1/\phi$. From (23) we have $EY \leq s+1$. Let $\beta = Mu(q_{s+1})$ where $M \geq 1$. Suppose that the second constraint is satisfied, i.e. (24) $= EY(w_f(1) - w_f(n) + nw_f(n) > Mu(q_{s+1}) - nw_f(n-s)$. ¿From claim 2 we have

$$EY > \frac{Mnw_f(n) + Mp_{s+1}d_f((n-s-1)(s+1) + (s+1)s) - nw_f(n)}{(n-1)d_f} =$$

$$\frac{n(M-1)w_f(n) + Mp_{s+1}d_f(s+1)(n-1)}{(n-1)d_f} \geq$$

$$\frac{M(s+1)}{\phi}$$

where the last inequality follows from the fact that $M \geq 1$ and that $p_{s+1} \geq 1/\phi$. Every $M \geq \phi$ will contradict that $EY \leq s+1$.

We showed that if $\beta = \phi max\{u(\pi_s), u(q_{s+1})\}$ then system $\widehat{P_1}$ doesn't have a solution. Therefore $\widehat{D_1}$ has a feasible solution with $\beta = \phi max\{u(\pi_s), u(q_{s+1})\}$, and therefore the surplus at every correlated equilibrium is not more than $\phi max\{u(\pi_s), u(q_{s+1})\}$. □

Proving that ϕ is an upper bound is highly non-trivial. Unfortunately, we do not know what is the least upper bound. However, the example below shows that the mediation value can be at least $\frac{9}{8}$.

Example 1: Let $n = 3$, $M = \{f, g\}$, $w_f = (24, 12, 0)$ and $w_g = (8, 8, 8)$. It can be shown that $v_N(\Gamma) = 32$, and it can be obtained, both in a pure-strategy equilibrium (two players choose f and the other player chooses g) and in a mixed- strategy equilibrium. Consider the following correlated strategy μ. Assign the probability $\frac{1}{6}$ to each strategy profile in which, not all players choose the same facility. μ is in equilibrium and the surplus at μ is 36. □

The above study shows that correlation is extremely helpful in the context of (even non-increasing) congestion games. We next show that correlation is helpful even in the narrow class of facility symmetric forms with non-increasing facilities.

Theorem 11. *$MV\left(\{\Gamma_F | F \in \mathcal{IN}_{n\times 2}\}\right) > 1$ for every $n \geq 4$.*

Proof of theorem 11
Consider the following family of forms. For every $j \in M \quad w_j = (10n, 1, .., 1, 1-\epsilon, 0)$ where ϵ is small ($n \geq 4$). Set $\pi = (1, n-1)$. Note that the maximal surplus is obtained by any strategy profile in A_π. Observe the following correlated strategy. Every strategy profile in A_π is played with probability $\frac{1}{2n}$. Note that there are exactly $2n$ strategy profiles in A_π. We have $g(1) = 10n$, $g(2) = 1$, $g(n-1) = 1-\epsilon$ and $g(n) = 0$. Therefore the incentive constraints (1) are: $\frac{10n+(1-\epsilon)-1)(n-1)}{2n} = \frac{(10n)-\epsilon(n-1)}{2n} \geq 0$. Hence the above correlated strategy is a correlated equilibrium and clearly obtains the maximal surplus. It remains to see

that no equilibrium (Nash) obtains the maximal surplus. Every strategy profile in A_π is not in equilibrium since every player that chooses the facility chosen by another $n-2$ players is willing to deviate. Otherwise if one or more players play both strategies with a positive probability then at least one strategy profile that does not obtain the maximal surplus will be played with a positive probability. $\square$

The case of symmetric forms with non-increasing facilities is quite restricting one. As a result, the fact the mediation value may be greater than 1 in this case is quite encouraging. However, if we further restrict the setting to obey some concavity requirements, the above does not hold any more. Formally, we say that a function $v : \{1, 2, ..., N\} \to \mathbb{R}^+$ is *concave* if for every integer $k \geq 2 \quad v(k+1) - v(k) \leq v(k) - v(k-1)$. We can now show:

Theorem 12. *Let $F \in \mathcal{IN}_{n\times m}$ and assume $n \geq m$. Define v by $v(k) = kg(k)$. If v is concave then there exists an equilibrium in Γ_F which obtains the maximal surplus.*

Proof of theorem 12
Let $\pi = (\pi_1, \pi_2, ..., \pi_m)$ be a congestion vector and let $j, l \in M$. First observe that $u(\pi) = \sum_{i=1}^{m} v(\pi_i)$. Let π^* be the same congestion vector as π only π_j and π_l are replaced by $\pi_i^* = \lfloor \frac{\pi_j+\pi_l}{2} \rfloor$ and $\pi_j^* = \lceil \frac{\pi_j+\pi_l}{2} \rceil$ respectively. We show that $u(\pi_*) \geq u(\pi)$. It is enough to show that $v(\pi_j^*) \geq v(\pi_l^*)$. But this follows directly from the concavity of v (recall that g is non-increasing). Set $k_1 = \lfloor \frac{n}{m} \rfloor$ and $k_2 = \lceil \frac{n}{m} \rceil$. Let π^o be a congestion vector where each coordinate is either k^1 or k^2. Assume there exist a congestion vector π that obtains the maximal surplus and such that not all it's coordinates are k_1 or k_2. In this case there exist two coordinates j, l that are not k_1 or k_2. Construct π^* as above and let $\pi = \pi^*$. Continue this process until $\pi = \pi^o$. $u(\pi^o)$ obtains the maximal surplus. Obviously π^o is in equilibrium. $\square$.

4.2 The Enforcement Value

We already know that the enforcement value is unbounded on the class of Prisoner's Dilemma games (notice that a Prisoner's Dilemma game is a simple congestion game). In addition, we show:

Theorem 13. *$sup\{EV(\Gamma)|F \in \mathcal{SN}_{3\times 2}$,
there are no strictly dominant strategies$\} = \infty$.*

The above result shows that the enforcement value may be unbounded already on the class of simple congestion games with non-increasing facility payoff functions. It turns out that it is unbounded even when restricting the games to those who are generated by symmetric congestion forms with non-increasing facility payoff functions:

Theorem 14. $\lim_{n\to\infty} EV\left(\{\Gamma_F | F \in \mathcal{IN}_{n\times 2}\}\right) = \infty$.

Proof of theorem 14
Consider the following family of forms F_n. Let $M = \{a_1, a_2\}$ and let $w_{a_1} = w_{a_2} = (\sqrt{n}, 1, 0, 0, .., 0)$. Recall that n is the number of players. First, observe that the congestion vector $\pi_n = (1, n-1)$ obtains the maximal surplus $\sqrt{n}$ (for every $n \geq 3$). It is enough to show that $v_C(\Gamma_{F_n})$ is bounded for every $n \geq 3$. We use the the dual program $(\widehat{D})$ to prove this result. Let (α, β) be a feasible solution for the dual problem then by the weak duality theorem $\beta \geq v_C(\Gamma_{F_n})$. Note that for each player $i \in N$, there are two dual variables $\alpha^i(a_1|a_2)$ and $\alpha^i(a_2|a_1)$. We will find a feasible solution where all dual variables except β have the same value x, i.e. for each player $i \in N, x = \alpha^i(a_1|a_2) = \alpha^i(a_2|a_1)$ and $\alpha^i = \alpha^k \quad \forall i, k \in N$. The dual program reduces to:

$$\begin{aligned}
&\min \beta \\
&\text{s.t.} \\
&x \geq 0, \\
&(-\sqrt{n} + n - 1)x + \beta \geq \sqrt{n}, \\
&-2x + \beta \geq 1, \\
&n\sqrt{n} + \beta \geq 0.
\end{aligned}$$

Set $x = 1$ and $\beta = 3$. Hence the dual problem is bounded from above by 3 for every $n \geq 3$ whereas the maximal surplus approaches infinity as $n \to \infty$. $\square$

Although the enforcement value may be unbounded when we have facility symmetric congestion forms, it is of great interest to characterize general cases of this natural setup where the correlation enables to get close to the maximal value. More specifically, we now characterize the cases where correlation allows to actually obtain the related maximal value, i.e. the enforcement value is 1.

The following characterization makes use of the following definition and notations. Let F be a simple congestion form with n players and m facilities. A congestion vector $\pi = \pi(n, m)$ is an m-tuple $\pi = (\pi_j)_{j \in M}$, where $\pi_1, \pi_2, ..., \pi_m \in Z^*$ (nonnegative integers) and $\sum_{j=1}^{m} \pi_j = n$. π represents the situation where π_j players choose facility j. Every strategy profile $A \in X$ uniquely determines a congestion vector π^A. Note that there are $\binom{n}{\pi_1}\binom{n-\pi_1}{\pi_2} \cdots \binom{n-\sum_{j=1}^{m-2} \pi_j}{\pi_{m-1}}$ strategy profiles in the game Γ_F that correspond to a congestion vector π, and denote by B_π the set of all such strategy profiles. Thus $B_\pi = \{A \in X | \, \pi^A = \pi\}$. Given a congestion vector π, all strategy profiles in B_π have the same surplus which we denote by $u(\pi)$. Therefore $u(\pi) = \sum_{j \in M} \pi_j w_j(\pi_j)$ where $w_j(0)$ is defined to be zero for every $j \in M$. We will say that a congestion vector π is in equilibrium if every strategy profile in B_π is in equilibrium. Let $\tau : M \to M$ be a one to one function and let $\tau\pi = (\tau\pi)_{j \in M}$ be the congestion vector defined by $(\tau\pi)_j = \pi_{\tau(j)}$. Let $F \in \mathcal{I}_{n \times m}$. In this case $u(\pi) = u(\tau\pi)$. Let $A_\pi = \bigcup_\tau B_{\tau\pi}$. Observe that $B_\pi \subseteq A_\pi$. Both sets are finite and therefore their elements can be ordered.

Theorem 15. *Let $F \in \mathcal{I}_{n \times m}$. Then $v_C(\Gamma_F) = opt(\Gamma_F)$ if and only if there exist a congestion vector $\pi = (\pi_1, ..., \pi_m)$ and a correlated equilibrium $\mu \in C(\Gamma_F)$ such that:*

1. $u(\pi) = opt(\Gamma_F)$.
2. μ *is distributed uniformly over all elements (strategy profiles) in* A_π.

Proof of theorem 15
Obviously, if there exist such a congestion vector π and such a correlated equilibrium $\mu \in C(\Gamma_F)$, then $v_C(\Gamma_F) = opt(\Gamma_F)$.

Before proving the other direction we need to make a few observations. Let $A \in X$. Denote by $\gamma^i(A)$ the facility player i chooses in A. Let π_k^A be the number of players that choose facility k in A.
For every $\pi = (\pi_1, ..., \pi_m)$ let $s_\pi(j) = \binom{n-1}{\pi_j - 1} \prod_{l \neq j} \binom{n-\pi_j-\sum_{k=1}^{l-1} \pi_k 1_{k\neq j}}{\pi_j}$ and let $D_{\pi,i,j} = \{A : A \in B_\pi \quad, \gamma^i(A) = j\}$. $D_{\pi,i,j}$ is the set of all strategy profiles that correspond to π and player i chooses facility j. Observe that $s_\pi(j) = |D_{\pi,i,j}|$ for every $i \in N$. Notice that $|A_\pi| = |B_\pi| m! = \binom{n}{\pi_1}\binom{n-\pi_1}{\pi_2} \cdots \binom{n-\sum_{j=1}^{m-2}\pi_j}{\pi_{m-1}} m!$.

Define $Z(\pi)$ as follows:

$$Z(\pi) = \frac{\sum_{j=1}^m s(j) \sum_{k\neq j}(g(\pi_j) - g(\pi_k + 1))}{|A_\pi|} =$$

$$= \frac{\sum_{j=1}^m \pi_j \sum_{k\neq j}(g(\pi_j) - g(\pi_k + 1))}{m!n} \tag{27}$$

since $\frac{\binom{n}{k-1}}{\binom{n}{k}} = \frac{k}{n}$.

Let $\pi = (\pi_1, ..., \pi_m)$ be a congestion vector such that $u(\pi) = opt(\Gamma_F)$ and let μ be a correlated strategy distributed uniformly over all elements in A_π. Observe that μ is a correlated equilibrium if and only if $Z(\pi) \geq 0$.

Let $D = \bigcup_{\pi:u(\pi)=opt(\Gamma_F)} A_\pi$. Let $J = |D|$. Define the matrix $A_{J\times n(m^2-m)}$ as follows:

$$A(d, ijk) = \frac{[g(\lambda^i(d)) - g(\pi_k^d + 1)]1_{ij}(d)}{2n} \tag{28}$$

where each row corresponds to a strategy profile $d \in D$ and column ijk corresponds to player i who chooses jth strategy (facility) and deviates to the kth strategy ($j \neq k$), and:

$$1_{ij}(d) = \begin{cases} 1 \ \gamma^i(d) = j \\ 0 \ otherwise. \end{cases}$$

We prove the other direction. Assume $v_C(\Gamma_F) = opt(\Gamma_F)$ and let $\xi \in C(\Gamma_F)$ such that $u(\xi) = v_C(\Gamma_F)$. Assume by contradiction that there doesn't exist a congestion vector π and a correlated equilibrium $\mu \in C(\Gamma_F)$ such that μ is distributed uniformly over all elements in A_π and $u(\pi) = opt(\Gamma_F)$. Let $\pi = (\pi_1, ..., \pi_m)$ be a congestion vector such that $u(\pi) = opt(\Gamma_F)$. Since there is no $\mu \in C(\Gamma_F)$ that is distributed uniformly over all elements in A_π it must be that $Z(\pi) < 0$. We use lemma 3. Every row d (in the matrix A) corresponds to strategy profile $d \in D$ which itself corresponds to a congestion vector $\pi^d = (\pi_1^d, ..., \pi_m^d)$. Set $b(d) = Z(\pi^d)$. The vector $x = (1, 1, ..., 1)$ satisfies (i) in lemma 3, since the left hand side is exactly $Z(\pi^d) \quad \forall d \in D$. Therefore, system (ii) doesn't have a solution. In particular if μ is scaled so that it is a probability distribution, every inequality in system (ii) is exactly an incentive constraint that defines a correlated equilibrium under the assumption that positive probabilities can be assigned only to strategy profiles in D. Therefore $u(\xi) = v_C(\Gamma_F) < opt(\Gamma_F)$ which is a contradiction to the assumption that $v_C(\Gamma_F) = opt(\Gamma_F)$. $\square$

The proof of the above result shows implicitly the existence of many situations where correlation allows to obtain the maximal surplus, while (without correlation) the best mixed-strategy equilibrium behaves poorly. We demonstrate this by the following example.

Example 2: Let $F \in \mathcal{I}_{6\times 2}$. Let $w_j = (1.5, 1, 4, 4.5, 4.5, 3)$ for every $j \in M$. The maximal surplus is obtained in any strategy profile that belongs to A_{π_1} and A_{π_2} where $\pi_1 = (3, 3)$ and $\pi_2 = (1, 5)$. It is easy to see that the both π_1 and π_2 are not in equilibrium. Let ξ_1 and ξ_2 be correlated strategies that are uniformly distributed over A_{π_1} and A_{π_2} respectively. In order to check if there exists a correlated equilibrium that obtains the maximal surplus it is enough to check whether ξ_1 or ξ_2 are correlated equilibria. Indeed, one can easily check that ξ_2 is a correlated equilibrium whose surplus equals the maximal surplus. $\square$

5 Conclusion

In this paper we have introduced and studied two measures for the value of correlation in strategic interactions: the mediation value and the enforcement value. These measures complement existing measures appearing in the price of anarchy literature, which are comparing the maximal surplus (when agent behavior can be dictated) to the surplus obtained in Nash equilibrium (when agents are selfish). Indeed, correlation captures many interesting situations, which are common to distributed systems and computer science applications. In many systems a reliable party can advise the agents on how to behave but can not enforce such behavior. The gain that may be obtained by this capability is the major subject of the study presented in this paper. We have studied and shown the power of this approach both for general games and in the context of congestion games.

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